Niven Ring Gravitational Stability

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Abstract. A “Niven ring” is a ring of material placed around a star so as to provide a very large inhabitable region. The ring would be solid, about a million miles wide and would be at about earth’s orbit, around 100 million miles from the sun. The ring would be set spinning so that it would provide an equivalent to earth’s gravity field by the action of centrifugal force. Walls a thousand miles high along the edges hold in the atmosphere.

The origin of the Niven ring idea is L. Niven’s 1970 novel “Ringworld”. Sometime after writing this book, the author was informed that a Niven ring would be gravitationally unstable. This resulted in further plot development by the author in his 1980 novel, “The Ringworld Engineers”, which describes the efforts necessary to keep the Niven ring orbiting stably.

In this short paper we derive the gravitational instability of a Niven ring without calculus. We show that the ring orbit decays exponentially with a time period τ of about two months.

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1. INTRODUCTION

Larry Niven’s book “Ringworld” [1] describes what is now known as a “Niven ring”, a ring of material surrounding a star at a comfortable (for life) distance, and set in motion so as to provide centrifugal acceleration equivalent to earth’s gravitational field.

The orbit of a planet in the gravitational field of the sun is stable, that is, if we make small changes to the orbital parameters this will result in only small changes to the orbit of the planet. For example, if the earth is hit and absorbs a small meteor it will continue to orbit the sun with quite nearly the same period and at the same distance.

Surprisingly, a Niven ring does not have this property; it is unstable. When Niven attended the 1971 World Science Fiction Convention, students from MIT changed “The Ringworld is unstable! The Ringworld is unstable!” [2] and so one reason he wrote “The Ringworld Engineers” [2] was to address this and other engineering problems.

After studying physics my first year in college I decided to major in mathematics. I began grad school in math about the time that “The Ringworld Engineers” was published and naturally I calculated its instability as a matter of personal interest. I enjoyed the calculation and this enjoyment was one of my motivations for deciding to go to physics grad school even though I didn’t have an undergraduate degree in physics. So part of the purpose of this paper is to share the enjoyment of an interesting calculation with my ITT Technical Institute physics students, and at the same time provide an example of a physics paper.

2. GRAVITATIONAL POTENTIAL ENERGY

The potential energy of a small object of mass m a distance r from the sun is given by

$$U = -G\frac{mM}{r} \tag{1}$$

where $G = 6.67428 \times 10^{-11}\text{m}^3\text{kg}^{-1}\ \text{s}^{-2}$. is the gravitation constant [3] and $M = 1.9891 \times 10^{30}\text{kg}$ is the mass of the sun. [4]

We are interested in the gravitational potential energy of objects around the orbital radius of the earth, that is, we wish to have $r = r_e \approx 1.496 \times 10^{11}\text{m}$. [5] Let's compute the gravitational potential energy for the mass m at two different distances, once at r_e, and again at a distance one meter greater: $r_e + 1\text{m}$. These two potential energies will be very close, to distinguish between them we will have to use a lot of digits. Plugging r_e and $r_e + 1\text{m}$ into Eq. (1), we find:

$$\begin{aligned} U_r &= -887420477.8074866\, m\,\text{J}, \\ U_{r+1} &= -887420477.8015547\, m\,\text{J} \end{aligned} \tag{2}$$

To move a mass m from r to $r+1$ requires that we do work against the gravitational potential of

$$U_{r+1} - U_r = 0.0059319m\,\text{J} \tag{3}$$

Since $W = Fx$, we can find the force necessary to do that work by dividing by the distance (one meter) to give:

$$F_r = 0.0059319m\,\text{N}. \tag{4}$$

The above force (which is necessary to cancel the gravitational force) is positive, therefore the gravitational force is negative. That is, the force of gravity tends to reduce the radius.

From $F = ma$ we get the acceleration of objects in the Sun's gravitational field (at the Earth's orbit) by dividing the above by m to find:

$$a_r = -0.0059319\ \text{ms}^{-2}. \tag{5}$$

These calculations show how we compute forces and accelerations from potential energies.

3. THE FORCE ON THE RING

The force we calculated in the previous section applies to only a small mass. Since force is a vector quantity, in computing the force on the Niven ring as a whole we would have to mentally break it into a large number of small pieces, compute the force on each piece, and then sum the forces up.

Since potential energy is a scalar quantity it is easier to add up. Consequently, to compute the force on the Niven ring we will break the ring into a large number of small pieces, add up the potential energies for each piece, and compute the force from the resulting potential energy. This will be a repeat of the previous section's calculation except that instead of applying the gravitational potential energy Eq. (1) to a single mass, we will apply it to a very large number of masses.

We choose a coordinate system with the center of the Niven ring at its center. We approximate the Niven ring with N pieces, equally arrayed around the origin at a distance r_e. Then the nth point has position:

$$(x_n, y_n) = r_e(\cos(2\pi n/N), \sin(2\pi n/N)). \tag{6}$$

If the sun is at the center of the Niven ring, then the total gravitational potential energy of the Niven ring is:

$$U_0 = -\Sigma_{n=1}^{n=N} GmM/r_e. \tag{7}$$

On the other hand, if the sun is offset by a distance of x meters in the $+x$ direction, the distance to the sun is no longer r_e for all the N points. Instead, we have to use the Pythagorean theorem to find the distance between the sun at $(x,0)$ and the nth point on the Niven ring at (x_n, y_n). By Eq. (6) we have:

$$\begin{aligned} r &= \sqrt{(x-x_n)^2+(0-y_n)^2}, \\ &= \sqrt{(x-r_e\cos(2\pi n/N))^2+(0-r_e\sin(2\pi n/N))^2}, \\ &= \sqrt{(x^2-2xr_e\cos(2\pi n/N)+r_e^2\cos^2(2\pi n/N)+r_e^2\sin^2(2\pi n/N)}, \\ &= \sqrt{(x^2-2xr_e\cos(2\pi n/N)+r_e^2}, \\ &= r_e(1-2(x/r_e)\cos(2\pi n/N)+(x/r_e)^2)^{1/2}. \end{aligned} \tag{8}$$

Thus the potential energy when the sun is at a distance of x meters is:

$$U_x = -\Sigma_{n=1}^{n=N}(GmM/r_e)(1-2(x/r_e)\cos(2\pi n/N)+(x/r_e)^2)^{-1/2}. \tag{9}$$

To compute the force when the sun is at x we do as in the previous section. We take the difference in potential energies between the sun at $x = x+0$ and the sun at $x+1$m and divide by the distance (one meter) to find:

$$\begin{aligned} F_x &= (U_{x+1}-U_{x+0})/1\text{m}, \\ &= -\Sigma_{n=1}^{n=N}(GmM/r_e)(1-2((x+1)/r_e)\cos(2\pi n/N)+((x+1)/r_e)^2)^{-1/2} \\ &\quad +\Sigma_{n=1}^{n=N}(GmM/r_e)(1-2((x+0)/r_e)\cos(2\pi n/N)+((x+0)/r_e)^2)^{-1/2}. \end{aligned} \tag{10}$$

The above can be simplified. First, factor out the GmM/r_e which appears in both summations. Then combine the two summations into one. And abbreviate $f_n(x) = 1-2(x/r_e)\cos(2\pi n/N)+(x/r_e)^2$. Then we have:

$$\begin{aligned} F_x &= -(GmM/r_e)\Sigma_{n=1}^{n=N}\left(\frac{1}{\sqrt{f_n(x+1)}}-\frac{1}{\sqrt{f_n(x)}}\right), \\ &= -(GmM/r_e)\Sigma_{n=1}^{n=N}\frac{\sqrt{f_n(x)}-\sqrt{f_n(x+1)}}{\sqrt{f_n(x+1)f_n(x)}}. \end{aligned} \tag{11}$$

Clearly, by symmetry, the force will be zero when $x = 0$. But we're interested in the force for small values of x. By "small" we mean small compared to $r_e = 1.496\times10^{11}$m.

And there's nothing special about "1 meter"; our program gives too much round-off error at that distance; we had to increase the distance to 1000 meters to get decent results.

The Java code for the function $f_n(x)$ is:

```
double fn(double Angle, double Xx, double Re) {
   double ret =   1.0
               - 2.0*(Xx/Re)*Math.cos(Angle)
               + (Xx*Xx)/(Re*Re);
   return ret;
}
```

and the program to compute Σ is:

```
double Re = 1.496E+11;
System.out.println(" Re = "+Re);
double Num = 100000.0; int NC = 100000;
System.out.println(" Num = "+Num);
for (double Xx = -100000.0;
   Xx < +100000.5; Xx = Xx+10000.0) {
   double DAngle = 0.0;
   double Sigma = 0.0;
   for (int I=0; I<NC; I++) {
      double X1 = Xx + 1000.0;
      double Fnx = Math.sqrt(fn(DAngle,Xx,Re));
      double Fn1 = Math.sqrt(fn(DAngle,X1,Re));
      Sigma = Sigma+(Fnx-Fn1)/(Fnx*Fn1);
      DAngle = DAngle + 2.0*Math.PI / Num;
   }
   Sigma = Sigma / 1000.0;
   System.out.println(Xx+" "+Sigma);
}
```

The program results are:

```
 Re = 1.496E11
 Num = 100000.0
-100000.0 -2.2272612780283174E-13
-90000.0 -2.0017987634905713E-13
-80000.0 -1.775813312753711E-13
-70000.0 -1.5493482795072766E-13
-60000.0 -1.3229176564945844E-13
-50000.0 -1.0971709098229928E-13
-40000.0 -8.702917605004044E-14
-30000.0 -6.448203694822028E-14
-20000.0 -4.7441270481899775E-14
-10000.0 -2.20105521630444E-14
0.0 6.438869112699861E-15
10000.0 2.3161224484774972E-14
```

```
20000.0 4.2964446118285605E-14
30000.0 6.839405651221105E-14
40000.0 9.109551706407309E-14
50000.0 1.1360579836542027E-13
60000.0 1.3072061917443282E-13
70000.0 1.5890091600418028E-13
80000.0 1.8154409098528625E-13
90000.0 2.041431870531509E-13
100000.0 2.2115586926925922E-13
```

The left column of the above gives the position x. The right column gives the sum Σ. It can be approximated by

$$\Sigma(x) = 2.2 \times 10^{-18} x \tag{12}$$

Plugging this into Eq. (11) we have:

$$F_x \;=\; -2.2 \times 10^{-18}\,(GmM/r_e)x. \tag{13}$$

The above is the force required to do the work against the gravitational potential. To get the force of gravity on the Niven ring, we take the negative of the above:

$$F_x \;=\; +2.2 \times 10^{-18}\,(GmM/r_e)x. \tag{14}$$

This is a spring force (i.e. Hooke's Law), but it is directed opposite to how a spring force is directed. That is, when $x > 0$, the force is positive and tends to make x even larger. Therefore the gravitational force on the Niven ring is unstable.

We can compute the acceleration on the Niven ring at the position x. To do this, we use $F = ma$. Note that the computer program assumed that $N = 100000$ so that the total mass of the Niven ring is $100000m$. Then:

$$\begin{aligned} a &= F_x/(100000m) = 2.2 \times 10^{-23}\,(GM/r_e)\,x, \\ &= 2.0 \times 10^{-14}\,x\,\mathrm{s}^{-2} \end{aligned} \tag{15}$$

When x is small, say one meter, the above acceleration is very small. The problem is that if the Niven ring is at all off-center, x will grow exponentially.

4. EXPONENTIAL GROWTH

When the ring position is one meter, Eq. (15) shows that it will feel an acceleration of 2.0×10^{-14} m s^{-2}. The acceleration at two meters will be twice this, so the average acceleration over the interval from 1 meter to 2 meter will be 3.0×10^{-14}. This is not an exact calculation, but we can estimate how long the ring will take to reach the 2 meter point, assuming it begins at 1 meter, by using the equation $T = \sqrt{2x/a}$:

$$T_{1\to 2} = \sqrt{2 \cdot 1/3.0 \times 10^{-14}} = 8.2 \times 10^{6}\ \mathrm{s}. \tag{16}$$

This is a little over 3 months. Similarly, if the ring starts off with a position of 2 meters, we can estimate the time required for the position to reach 4 meters. The initial acceleration is 4.0×10^{-14} and final acceleration twice this. Using the same formula as above, but now for a distance of 2 meters and the new average acceleration of 6.0×10^{-14} we find an approximate time of:

$$T_{2\to4} = \sqrt{2\cdot 2/6.0\times10^{-14}} = 8.2\times10^{6}\text{ s}. \tag{17}$$

This is identical to the time for getting to 2 meters from 1 meter. Similarly, the time for the position to grow from 4 meters to 8 meters is also a little over 3 months.

When we say that something grows exponentially over time t, we mean that it has the form

$$g(t) = g_0 e^{t/\tau}. \tag{18}$$

We can figure out τ by using our calculation Eq. (16) for the time required for the ring to double its position:

$$\begin{aligned} 2 &= g(t)/g_0 = \exp(8.2\times10^{6}/\tau), \text{ therefore} \\ \tau &= 8.2\times10^{6}\ln(2) = 5.7\times10^{6}\text{ s}, \end{aligned} \tag{19}$$

which is about 66 days.

The above analysis is unsatisfactory in that we have assumed that the ring was stationary at the beginning of one of the time periods, but it is not stationary at the end. To correct this, we will redo the calculation assuming an initial velocity v_0. Then we require that the final velocity be $2v_0$ just as the final position is twice the initial position. With an average acceleration $\bar{a}$ and a doubling time T we want:

$$\begin{aligned} 2v_0 &= v_0 + \bar{a}T, \\ v_0 &= \bar{a}T. \end{aligned} \tag{20}$$

Then the position calculation becomes:

$$\begin{aligned} 2x_0 &= x_0 + v_oT + (\bar{a}/2)T^2, \\ x_0 &= \bar{a}T^2 + (\bar{a}/2)T^2, \\ &= 1.5\bar{a}T^2. \end{aligned} \tag{21}$$

Since $\bar{a} = 2.0\times10^{-14}(x_0+2x_0)/2 = 3.0\times10^{-14}x_0$ we have

$$\begin{aligned} x_0 &= 1.5\times3.0\times10^{-14}x_0T^2, \text{ and so} \\ T &= 4.7\times10^{5}\text{ s} = 55\text{ days}. \end{aligned} \tag{22}$$

Thus taking into account the speed-up in velocity, the doubling time is slightly smaller than our rough calculation.

5. CONCLUSION

We've shown that the Niven ring is gravitationally unstable by computing its gravitational potential. Furthermore, we find that the time constant τ for the ring orbit to decay is 55 days. Since this time constant is relatively small, the stability of a Niven ring cannot be ignored for long periods of time.

Engineers who control a Niven ring will have to apply feedback to the ring to keep it in position. The frequency they will need to apply the feedback will depend on (a) how far off balance they are willing to let the ring drift, and (b) how close to zero they are able to manipulate the ring. In any case, the time constant for the ring is on the order of several months and the feedback mechanism needs to take this into account.

REFERENCES

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