Foundations of Physics manuscript No.
(will be inserted by the editor)

Carl Brannen

Quantum Information of Unitary Mixing Matrices

Received: 7/1/2010 / Accepted: date

Abstract Unitary mixing matrices, such as the quark and lepton mixing matrices, have unphysical degrees of freedom; one can multiply rows and columns by arbitrary complex phases. Industry practice is to choose phases and parameterize what's left of the unitary matrix. We show that one can use an unbiased state to define the complex phases; this defines a Lie subgroup of $U(n)$ that is isomorphic to $U(n-1)$. Existence is related to, but simpler than, the mutually unbiased bases (MUB) problem of quantum information theory. We show that Jarlskog invariants and J_{CP} are related to Berry-Pancharatnam or quantum phases.

For 3×3 matrices, we solve the relationship between the Lie subalgebra and Lie subgroup in closed form. This defines a new parameterization that is similar to the standard 3×3 parameterization used for the CKM matrix, but treats equally the (12), (23), and (13) permutations. The new fourth parameter gives the (123) and (132) permutations.

For the same Lie subgroup, we provide two simpler parameterizations. One of these brings the tribimaximal form for the MNS matrix into a particularly simple form.

Keywords CKM · MNS · Majorana · Jarlskog · tribimaximal · parameterization · Lie subalgebra · Lie subgroup · unitary · mixing · quark · lepton · neutrino · mixing matrix

PACS 12.15.Ff · 02.20.Sv

The first three sections of the paper concerns unitary matrices in general: (1) the "magic" subgroup of unitary matrices $U(n)$; (2) the question of

Carl Brannen
ITT Technical Institute,
1615 75th Street SW,
Everett, WA 98203, USA
E-mail: carl@brannenworks.com

whether any unitary matrix can be brought into magic form, and (3) the relationship between CP violation, Jarlskog invariants, and Berry-Pancharatnam phase. Sections four through six deal with parameterizations of 3×3 unitary mixing matrices: (4) the standard parameterization; (5) the permutation parameterization; and (6) two related parameterizations. Sections seven and eight cover applications to the CKM and MNS matrices, respectively. Section nine is a brief conclusion.

1 Magic Unitary Matrices

For brevity, we will write the pure density matrix for the quantum state a_j as simply

$$\hat{a}_j = |a_j\rangle\langle a_j|. \tag{1}$$

We will treat the pure density matrices as the fundamental objects. Given a complete set of states $\{a_j\}$, we have equations

$$\begin{aligned} \hat{a}_j \hat{a}_k &= \delta_{jk} \hat{a}_k, \\ \Sigma_j \hat{a}_j &= 1, \end{aligned} \tag{2}$$

that correspond to the orthonormality and completeness relations of state vectors.

Given two complete sets of states $\{a_j\}$ and $\{b_k\}$, a unitary mixing matrix is defined by the complex numbers

$$u_{jk} = \langle a_j|b_k\rangle. \tag{3}$$

The bras and kets have arbitrary complex phases, so that under this definition, the unitary matrix will depend on these complex phases. Converting a unitary mixing matrix to a matrix of observables requires that we specify the arbitrary complex phases.

If we define a unitary matrix using pure density matrices rather than bras and kets, the arbitrary complex phases will be automatically eliminated. This will give us an observable unitary matrix.

Given a pure density matrix, we can obtain a bra or ket by taking any nonzero row or column, respectively, and normalizing. For a ket, picking a nonzero column amounts to multiplying the pure density matrix $\hat{a}_j$ on the right by a state vector $|v\rangle$, which has a one in a single location and all other entries zero.

It would be more convenient if we could choose $|v\rangle$ once for all the $\hat{a}_j$ and $\hat{b}_k$. The requirement of "picking a nonzero column" is equivalent to requiring

$$\langle a_j|v\rangle \neq 0 \;\text{ and }\; \langle b_k|v\rangle \neq 0. \tag{4}$$

For further convenience, we require that all the above inner products have the same magnitude. For example, given $\{\langle a_j|\} = \{(1,0),(0,1)\}$ and $\{\langle b_k|\} = \{(\cos(\theta),\sin(\theta)),(-\sin(\theta),\cos(\theta))\}$, we could choose $\langle v| = \sqrt{1/2}(1,i)$ and have all the inner products with magnitude $\sqrt{1/2}$. In general, for a Hilbert space of dimension n, we require

$$|\langle a_j|v\rangle|^2 = |\langle b_k|v\rangle|^2 = 1/n. \tag{5}$$

Does $|v\rangle$ always exist? We postpone this nontrivial question to the next section.

Assuming $|v\rangle$ exists, we have that

$$\begin{aligned} \sqrt{n}\langle v|a_j\rangle &= \exp(i\alpha_j), \\ \sqrt{n}\langle b_k|v\rangle &= \exp(i\beta_k), \end{aligned} \tag{6}$$

for real angles α_j, β_k. Then the following defines the elements of a unitary matrix:

$$\begin{aligned} v_{jk} &= \sqrt{n}\langle v|a_j\rangle \ \langle a_j|b_k\rangle \ \sqrt{n}\langle b_k|v\rangle, \\ &= n \ \mathrm{tr} \ (\ \hat{v} \ \hat{a}_j \ \hat{b}_k \ \hat{v} \ ). \end{aligned} \tag{7}$$

Since this definition uses only pure density matrices, it is free of unobservable complex phases.

Define an s-magic matrix (compare [1]) as one whose rows and columns sum to s. The unitary matrix $[v_{jk}]$ is 1-magic:

$$\Sigma_j v_{jk} = \Sigma_k v_{jk} = 1. \tag{8}$$

Multiplying an s-magic matrix by a t-magic matrix gives an st-magic matrix, so our choice of phase for the unitary matrices defines a magic Lie group, we will refer to them as $MU(n)$. The corresponding Lie algebra consists of the Hermitian 0-magic matrices, $mu(n)$.

There are $(n-1)^2$ real degrees of freedom in $mu(n)$. This suggests that they are isomorphic to the Hermitian $(n-1)\times(n-1)$ matrices, i.e. $u(n-1)$. To prove this, we need an Hermitian isomorphism which takes $mu(n)$ to a block diagonal form $(n-1)\times(n-1) \ + \ 1\times 1$.

An Hermitian isomorphism from X to Y that preserves multiplication and addition is

$$Y = HXH^{-1} \tag{9}$$

where H is Hermitian and unitary so $H = H^{-1}$. The Householder matrices are such; they are defined as:

$$H = 1 - 2|u\rangle\langle u| \tag{10}$$

where $|u\rangle$ is a unit vector. To block diagonalize $mu(n)$ we use:

$$\begin{aligned} |u\rangle &= (u_1, u_1, ...u_1, u_n), \\ u_1 &= \sqrt{(\sqrt{n}+1)/(2\sqrt{n}(n-1))}, \\ u_n &= -\sqrt{(\sqrt{n}-1)/(2\sqrt{n})}. \end{aligned} \tag{11}$$

Since the Householder transformation preserves multiplication and addition, it also defines an isomorphism between $MU(n)$ and $U(n-1)$.

2 MUBs and the Fictitious Vacuum

In the previous section we showed that we can convert a unitary matrix to 1-magic form if we can find a state with all transition probabilities equal, Eq. (5). In this section we relate this problem to a similar one widely researched in the quantum information theory (QIT) community.

Momentum and position are complementary observables. Perfect knowledge of one implies no knowledge of the other. QIT studies complementary observables in the context of n-dimensional (finite) Hilbert spaces. Given two complementary bases $\{c_j\}$ and $\{d_k\}$, QIT refers to the bases as "unbiased". Suppose we have a state c_1. Since we can have "no knowledge" about the other observable, we must have that all the transition probabilities are equal:

$$\mathrm{tr}\,(\hat{c}_1\hat{d}_j\hat{c}_1) = \mathrm{tr}\,(\hat{c}_1\hat{d}_k\hat{c}_1). \tag{12}$$

Since $\Sigma_j\hat{d}_j = 1$, each of the n^2 transitions probabilities is equal to $1/n$.

A quantum state has more information than can be extracted from a single measurement. For example, spin-1/2 particles having spin in the $+x$ direction cannot be distinguished from those having spin in the $-y$ direction by using a measurement of spin in the $\pm z$ direction. However, given a large number of particles all in the same (but unknown) state, it is possible to obtain information on the state by applying different measurements to the particles. This is called "quantum tomography". [2] For spin-1/2, the complete density matrix can be determined (approximated) by $3 = 2 + 1$ measurements; spin in the $\pm x$, $\pm y$, and $\pm z$ directions.

In terms of minimizing the number of measurements, the ideal condition for quantum tomography is to have $n+1$ bases which are pair-wise unbiased. [3, 4] This is the maximum number of "mutually unbiased bases" (MUBs) possible, and is called a "complete set of MUBs". Complete sets are most efficient at quantum tomography. They've been found for n a power of a prime but the general case is an important unsolved problem in quantum information theory.

In QIT terms, we are looking for a state that is mutually unbiased with respect to the two arbitrary bases $\{a_j\}$ and $\{b_k\}$. The problem appears to be simpler than, but related to, the problem of finding complete sets of MUBs. It seems possible that continuing work on the QIT problem will provide assistance on our problem or vice versa.

In the late 1950s, Julian Schwinger wrote down a version of quantum field theory that is now known as "Schwinger's Measurement Algebra". [5, 6] He defined a "primitive measurement" as a projection operator which cannot be written as the sum of two non zero projection operators and associated them with the elementary particles. These correspond roughly to pure density matrices. An example of a primitive measurement would be the Pauli spin projection operator for spin in the $+x$ direction, $M(+\mathbf{x}) = (1+\sigma_x)/2$. In terms of a physical experiment, the primitive measurements correspond to a piece of equipment which allows particles to pass only if they possess a particular set of quantum numbers. Since the projection operators are idempotent, an incoming particle of the same type as the measurement, is left unchanged by the measurement. His theory is elegant in that it puts the states and the observables on the same mathematical footing. The related Schwinger-Weyl construction shows that three MUBs always exist in any Hilbert space of dimension larger than one. [7]

In addition to the primitive measurements $M(a_j)$, Schwinger also assumed operators where the outgoing particle is in a different state from the incoming particle $M(a_j, b_k)$. An example would be a spin raising operator $M(+\mathbf{z}, -\mathbf{z})$.

In order to provide creation and annihilation operators, Schwinger assumed the existence of a "fictitious vacuum" v. This is a state which corresponds to no particle. When the fictitious vacuum is the incoming state, we have a creation operator; when it is the outgoing state, we have an annihilation operator:

$$\begin{aligned} a_j^\dagger &= M(a_j, v) = \sqrt{n}\;\hat{a}_j\;\hat{v}, \\ a_j &= M(v, a_J) = \sqrt{n}\;\hat{v}\;\;\hat{a}_j. \end{aligned} \tag{13}$$

Schwinger's work did not assume that the fictitious vacuum was in the same Hilbert space as the states. Of course it's impossible to find a state that is unbiased with respect to all the bases of a Hilbert space. Perhaps this is why Schwinger chose the term "fictitious".

For the case of any one particular unitary matrix, we have only two bases to consider and Schwinger's fictitious vacuum may exist in the Hilbert space. The mathematical requirements are that all the transition probabilities are equal.

We can always choose one of the bases defining a unitary matrix U as the standard basis. The general form of a state $|v\rangle$ unbiased with respect to the standard basis is a vector of phases:

$$|v\rangle = (e^{iv_1}, e^{iv_2}, ..., e^{iv_n})/\sqrt{n}, \tag{14}$$

where v_j are real numbers. There are n real degrees of freedom in such a vector. For this to be unbiased with respect to the other basis, we must also have:

$$U|v\rangle = (e^{iv_1'}, e^{iv_2'}, ..., e^{iv_n'})/\sqrt{n}. \tag{15}$$

Thus the $|v\rangle$ we desire is a vector of phases that is mapped by the unitary matrix to a vector of phases.

Each basis state b_k gives one real restriction on the general vector of phases:

$$\mathrm{tr}\;(\hat{b}_k\hat{v}) = 1/n. \tag{16}$$

There are n such basis state restrictions, but if the first $n-1$ are satisfied, the last is automatic since $\sum_j \hat{b}_j = 1$. Thus we expect to find one degree of freedom left over in $|v\rangle$, which is just the vector's arbitrary complex phase. This suggests that a typical case will involve a finite number of choices for $|v\rangle$. If the bases are related, for example identical, then there are likely to be an infinite number of choices for $|v\rangle$.

Computer calculation with randomly chosen unitary bases indicates that the number of choices for $|v\rangle$ is indeed finite. The number N of choices for $|v\rangle$ increases rapidly with dimension n:

$$\begin{array}{c|cccc} n & 1 & 2 & 3 & 4 \\ \hline N & 1 & 2 & 4 & 12 \end{array}. \tag{17}$$

It's easy to show that $N = 2$ when $n = 1$. The existence of solutions for $n = 3$ was settled by Philip Gibbs.[8]

While the construction of the 1-magic unitary matrices requires a state vector $|v\rangle$, it may be useful to generalize the problem to density matrices. The completely mixed density matrix, $\hat{1}/n$, is unbiased with respect to all other states, pure or not. Given two density matrices $\hat{v}_1$ and $\hat{v}_2$ both unbiased with respect to two bases, all the states between them are also unbiased:

$$\hat{v}_\alpha = \alpha\hat{v}_1 + (1-\alpha)\hat{v}_2. \tag{18}$$

Thus the solution set is convex.

3 CP Violation and Berry-Pancharatnam Phases

Any product of pure density matrices that begins and ends with the same pure state $\hat{x}$ is some complex number k times that pure state. We write:

$$\hat{x}\hat{y}\hat{z}...\hat{x} = k_{xyz...x}\ \hat{x} \tag{19}$$

where $\hat{x}, \hat{y}, \hat{z}...$ are pure density matrices and $k_{xyz...x}$ is a number. If the left hand side happens to be zero, we define k to be zero as well. The k are observables. For example, the transition probability between $\hat{x}$ and $\hat{y}$ is given by $|\langle x|y\rangle|^2 = k_{xyx}$.

When a quantum state is sent through a sequence of operations and then returns to its original state, it is possible for it to return multiplied by a complex phase. These are called "Berry-Pancharatnam phases" [9, 10] or "quantum phases" and are observables even though they are complex numbers. For k to have a non zero imaginary part, we must have two intermediate states.

For example, with σ_j the Pauli spin matrices, the following product of projection operators: [11]

$$\hat{z}\hat{y}\hat{x}\hat{z} = \frac{1+\sigma_z}{2}\ \frac{1+\sigma_y}{2}\ \frac{1+\sigma_x}{2}\ \frac{1+\sigma_z}{2}, \tag{20}$$

is a complex multiple of the projection operator $\hat{z} = (1+\sigma_z)/2$. The observable k_{zyxz} does not depend on the choice of representation of the spin matrices; the complex constant i arises from the algebra of the Pauli spin matrices through $i = \sigma_x\sigma_y\sigma_z$. Physically, the above defines the phase $\pi/4$, picked up by a particle which navigates its way through a sequence of four Stern-Gerlach experiments oriented in the $+z$, $+x$, $+y$, and $+z$ directions.

An important example of Berry-Pancharatnam phases in elementary particles are the complex phases picked up by a fermion which emits a series of W^+ and W^- bosons. An up, charm, or top quark $\{u, c, t\}$ can emit a W^+ and become a down, strange, or bottom quark $\{d, s, b\}$:

$$\{u, c, t\}\ \rightarrow\ W^+\ +\ \{d, s, b\}. \tag{21}$$

Similarly, the $\{d, s, b\}$ can emit a W^-:

$$\{d, s, b\}\ \rightarrow\ W^-\ +\ \{u, c, t\}. \tag{22}$$

The two types of quarks $\{d, s, b\}$ and $\{u, c, t\}$, define two bases for the 3-dimensional Hilbert space. The transition amplitudes define a unitary matrix known as the CKM matrix:

$$V_{CKM} = \begin{pmatrix} \langle u|d\rangle & \langle u|s\rangle & \langle u|b\rangle \\ \langle c|d\rangle & \langle c|s\rangle & \langle c|b\rangle \\ \langle t|d\rangle & \langle t|s\rangle & \langle t|b\rangle \end{pmatrix} \tag{23}$$

In elementary particles literature the CKM is defined with the weak force boson interaction included so $(u, c, t)^t = V_{CKM}\gamma^0(1-\gamma^5)/2\ (d, s, b)^t$. [12] Our abbreviation is the usual QIT liberty of ignoring force bosons; either way one obtains the same V_{CKM}.

To find Berry-Pancharatnam phases in V_{CKM} we must consider transitions between pairs of states such as $\{d, s\}$ and $\{u, c\}$. For example, the observable for the transition sequence $d \to c \to s \to u \to d$:

$$\begin{aligned} k_{duscd}\ \hat{d} &= \hat{d}\hat{u}\hat{s}\hat{c}\hat{d}, \quad \text{or} \\ k_{duscd} &= \langle d|u\rangle\langle u|s\rangle\langle s|c\rangle\langle c|d\rangle, \\ &= V_{du}\, V^*_{su}\, V_{sc}\, V^*_{dc}, \end{aligned} \tag{24}$$

where V_{jk} are the entries in the CKM mixing matrix. The k_{duscd} is a Jarlskog invariant. [13] Note that $k^*_{duscd} = k_{dcsud}$; complex conjugation reverses the ordering.

Since our states $\{u, c, t\}$ and $\{d, s, b\}$ do not include space or time dependence, the action of CP on them is reduced to the complex conjugate. Thus an observable measure of CP violation is the imaginary part of k_{duscd}, which we can write as a difference between two observables:

$$J_{CP} = (k_{duscd} - k^*_{duscd})/2 = (k_{duscd} - k_{dcsud})/2. \tag{25}$$

Since $\{d, s, b\}$ form a complete basis we have:

$$\hat{s} = 1 - \hat{d} - \hat{b}. \tag{26}$$

Substituting the above in Eq. (24) and Eq. (25) we have

$$\begin{aligned} k_{duscd} - k_{dcsud} &= (k_{du1cd} - k_{dudcd} - k_{dubcd}) - (k_{dc1ud} - k_{dcdud} - k_{dcbud}), \\ &= 0 - k_{dudcd} - k_{dubcd} - 0 + k_{dcdud} + k_{dcbud}, \\ &= k_{dcbud} - k_{dubcd}. \end{aligned} \tag{27}$$

Thus J_{CP} for transitions between $\{d, b\}$ and $\{u, c\}$ is equal to the J_{CP} for transitions between $\{d, s\}$ and $\{u, c\}$. More generally, J_{CP} is an invariant of the 3×3 CKM matrix, that is, it does not depend (except for sign) on the choice of pairs of states considered. And since we've written it in terms of pure density matrices, there is no dependence on the arbitrary complex phases of the rows and columns of the matrix. All CP violations in the quarks are proportional to J_{CP}.

4 The Standard Parameterization for U(3)

After eliminating the degrees of freedom one obtains by multiplying rows and columns by complex phases, the $n \times n$ unitary matrices have $(n-1)^2$ real degrees of freedom. Thus the 3×3 experimental unitary matrices require 4 parameters. The standard parameterization uses $\{\theta_{12}, \theta_{23}, \theta_{13}, \delta\}$:

$$\begin{array}{lll} c_{12} = \cos(\theta_{12}), & c_{23} = \cos(\theta_{23}), & c_{13} = \cos(\theta_{13}), \\ s_{12} = \sin(\theta_{12}), & s_{23} = \sin(\theta_{23}), & s_{13} = \sin(\theta_{13}), \end{array} \tag{28}$$

$$U_{\text{st.}} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \tag{29}$$

The experimentally convenient observables of the V_{CKM} consist of the transition probabilities and J_{CP}.

The $\theta_{12}, \theta_{23}, \theta_{13}$, parameters are "mixing angles", they correspond roughly to the probabilities that the generations change:

$$\begin{array}{l} \theta_{12} \to P_{dc},\ P_{su}, \\ \theta_{23} \to P_{st},\ P_{bc}, \\ \theta_{13} \to P_{dt},\ P_{bu}. \end{array} \tag{30}$$

The δ parameter is intended to define CP violation. The formula for J_{CP} is:

$$J_{CP} = \sin(\delta)\sin(\theta_{23})\sin(\theta_{13})\sin(\theta_{12})\cos(\theta_{23})\cos^2(\theta_{13})\cos(\theta_{12}). \tag{31}$$

Thus δ defines CP violation to the extent that if δ is zero, so is J_{CP}. However, setting any of the other three parameters to zero will also force $J_{CP} = 0$. Note that the above equation is almost symmetric in the mixing angles, but θ_{13} has an extra cosine factor.

A "democratic" unitary matrix has all amplitudes equal in magnitude. A symmetric example is the discrete Fourier transform matrix:

$$F_3 = \sqrt{1/3}\begin{bmatrix} e^{2i\pi/3} & e^{4i\pi/3} & 1 \\ e^{4i\pi/3} & e^{8i\pi/3} & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{32}$$

whose entries are, in general, $\sqrt{1/n}\exp(2i\pi jk/n)$. The 3×3 democratic mixing matrix happens to have the maximum possible $J_{CP} = \sqrt{3}/18$. It clearly has all mixing angles equal. This example suggests that it is more natural to treat J_{CP} as a function of the mixing angles rather than as an independent parameter.

5 Permutation Parameterizations

The usual solution to the problem of writing a subgroup of the 2×2 unitary matrices without the arbitrary complex phases is to use the real subalgebra:

$$A_\theta = \exp(i\theta \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{33}$$

The corresponding $MU(2)$ 1-parameter subgroup is:

$$B_{2\theta} = \exp(i\theta \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}) = \frac{1}{2} \begin{bmatrix} 1+e^{2i\theta} & 1-e^{2i\theta} \\ 1-e^{2i\theta} & 1+e^{2i\theta} \end{bmatrix}. \tag{34}$$

Since A_θ and $B_{2\theta}$ have entries with the same magnitudes, they correspond to the same physical mixing matrix. We see that the real $U(2)$ matrices are a double cover of the $MU(2)$ matrices.

To convert A_θ to $B_{2\theta}$ we use two phase matrices and an overall phase $\exp(i\theta)$:

$$B_{2\theta} = e^{i\theta} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} A_\theta \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix} \tag{35}$$

The Majorana phase is traditionally defined by twice the phase of the right hand phase matrix's bottom right component. In the above, that position has $\exp(-i\pi/2)$ so we see that the new Majorana phase (for 2×2 matrices) will be different from the traditional by π. Thus a calculation which obtains a Majorana phase of π such as that of Rodejohann [14] (who used the full 3×3 matrices), will now obtain zero.

Since $MU(n) \equiv U(n-1)$, we can parameterize $MU(3)$ by writing down a parameterization for $U(2)$ which, in turn, can be parameterized by exponentiating its Lie algebra, $u(2)$. An arbitrary Hermitian 2×2 matrix can be parameterized as real multiples of σ_x, σ_y, σ_z and the unit matrix. Its exponential follows immediately:

$$\begin{aligned} U(2) &= \exp(i \begin{bmatrix} w+z & x-iy \\ x+iy & w-z \end{bmatrix}) \\ &= e^{iw} \begin{bmatrix} \cos(u) + iz\sin(u)/u & i(x-iy)\sin(u)/u \\ i(x+iy)\sin(u)/u & \cos(u) - iz\sin(u)/u \end{bmatrix} \end{aligned} \tag{36}$$

where $u = \sqrt{x^2+y^2+z^2}$.

To convert the above $U(2)$ parameterization into a Lie algebra parameterization for $MU(3)$ we use the Householder transformation of Eq. (11), to solve:

$$MU(3) = \exp(i \begin{bmatrix} \theta_{12}+\theta_{13} & -\theta_{12}+i\theta_{123} & -\theta_{13}-i\theta_{123} \\ -\theta_{12}-i\theta_{123} & \theta_{12}+\theta_{23} & -\theta_{23}+i\theta_{123} \\ -\theta_{13}+i\theta_{123} & -\theta_{23}-i\theta_{123} & \theta_{13}+\theta_{23} \end{bmatrix}) \tag{37}$$

in closed form. We obtain:

$$
\begin{aligned}
MU(3) = \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} &+ e^{iw}\tfrac{\cos(u)}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \\
+ e^{iw}\tfrac{i\sin(u)}{u}\begin{pmatrix} \phi_{23} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{13} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{12} \end{pmatrix} &+ e^{iw}\tfrac{\sin(u)\theta_{123}}{u}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
w &= \theta_{12} + \theta_{23} + \theta_{13}, \\
\phi_{12} &= (\theta_{13} + \theta_{23} - 2\theta_{12})/3, \\
\phi_{23} &= (\theta_{12} + \theta_{13} - 2\theta_{23})/3, \\
\phi_{13} &= (\theta_{23} + \theta_{12} - 2\theta_{13})/3, \\
u &= \sqrt{3\theta_{123}^2 + 1.5(\phi_{12}^2 + \phi_{23}^2 + \phi_{13}^2)}.
\end{aligned}
\tag{39}
$$

A permutation matrix is a unitary matrix with all entries either zero or one. In the case of both the standard parameterization and the permutation parameterization, the three odd permutations (jk) are obtained by putting all the angles but θ_{jk} to zero. The permutation parameterization gives the even permutations (123) and (132) by arranging for all the angles but θ_{123} to be zero. Setting all the angles to zero gives the identity.

Solving for J_{CP} we find:

$$
\begin{aligned}
J_{CP} = &(\phi_{12}^3 + \phi_{23}^3 + \phi_{13}^3)\cos(w)\sin^3(u)/(3u^3) \\
&+ 2\theta_{123}^2\sin(w)\cos(u)\sin^2(u)/(3u^2) \\
&+ 2(\cos(w) - \cos^3(u))\sin(w)/27,
\end{aligned}
\tag{40}
$$

a symmetric function of the mixing angles.

The democratic unitary 3×3 mixing matrix is obtained, in the standard and the permutation parameterizations as follows:

$$
\begin{array}{r|ccccc}
 & \theta_{12} & \theta_{23} & \theta_{13} & \delta & \theta_{123} \\
\hline
\text{Standard:} & \pi/4 & \pi/4 & 0.61548 & \pi/2 & \\
\text{Permutation:} & 2\pi/9 & 2\pi/9 & 2\pi/9 & & 0
\end{array}.
\tag{41}
$$

The democratic permutation mixing angles are all equal and close to the average of the standard mixing angles.

6 Two More U(3) Parameterizations

An "m-circulant" matrix has each row identical to the row above, but rotated m to the right. It is easily shown that any 1-magic 3×3 matrix (unitary or otherwise) can be written as the average of three magic matrices: a 1-magic matrix with all entries equal to 1, a 1-circulant 0-magic matrix defined by its top row (A_1, A_2, A_3), and a 2-circulant 0-magic matrix with top row (B_1, B_2, B_3):

$$
\frac{1}{3}\begin{bmatrix} 1 + A_1 + B_1 & 1 + A_2 + B_2 & 1 + A_3 + B_3 \\ 1 + A_3 + B_2 & 1 + A_1 + B_3 & 1 + A_2 + B_1 \\ 1 + A_2 + B_3 & 1 + A_3 + B_1 & 1 + A_1 + B_2 \end{bmatrix}.
\tag{42}
$$

The above decomposition is unique and makes a natural target for a parameterization of $MU(3)$.

Let α, β, γ, and δ be four real numbers. Then:

$$\begin{aligned} A_n &= 2\cos(\gamma)\cos((\alpha+2n\pi)/3)e^{i\delta}, \\ B_n &= 2i\sin(\gamma)\cos((\beta+2n\pi)/3)e^{i\delta}, \end{aligned} \tag{43}$$

gives a parameterization of $MU(3)$. The Jarlskog invariant for this parameterization is:

$$J_{CP} = (\cos(2\delta) - 2\cos^3(\gamma)\cos(\alpha)\sin(\delta) + 2\sin^3(\gamma)\cos(\beta)\cos(\delta))/27. \tag{44}$$

This is simpler than the permutation parameterization but does not solve the 1-parameter subgroups of $MU(3)$.

Multiplying a 1-magic unitary matrix by $e^{-i\delta}$ gives a $e^{-i\delta}$-magic matrix. Such a transformation makes A_n pure real and B_n pure imaginary. After some algebra, this provides another parameterization of $MU(3)$:

$$U = \frac{1}{3}\begin{bmatrix} C_1 & C_2 & C_3 \\ C_3 & C_1 & C_2 \\ C_2 & C_3 & C_1 \end{bmatrix} + \frac{i}{3}\begin{bmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & D_1 \\ D_3 & D_1 & D_2 \end{bmatrix}, \tag{45}$$

where

$$\begin{aligned} C_n &= \cos(\delta) + 2\cos(\gamma)\cos((\alpha+2n\pi)/3), \\ D_n &= \sin(\delta) + 2\sin(\gamma)\cos((\beta+2n\pi)/3). \end{aligned} \tag{46}$$

The Jarlskog invariant becomes:

$$J_{CP} = (2\cos^3(\gamma)\sin(\delta)\cos(\alpha) + 2\sin^3(\gamma)\cos(\delta)\cos(\beta) - \sin(2\delta))/27. \tag{47}$$

The standard parameterization is organized around the assumption that the weak force acts primarily to leave the generation number alone, and secondarily to change the generation number by 1. This is in contrast to the above, which organizes changes to the generation number as if the generations were cyclic. [15]

7 The CKM Matrix

The CKM fitter group's 2009 estimates [16] for the modified Wolfenstein parameters are:

$$\begin{aligned} \lambda &= 0.2257, \quad A = 0.814, \\ \bar\rho &= 0.135, \quad \bar\eta = 0.349. \end{aligned} \tag{48}$$

These define the standard parameters $\{\theta_{12}, \theta_{13}, \theta_{23}, \delta\}$.

From computer calculation, we find that typical 3×3 unitary matrices can be converted to 1-magic form in four ways. Thus there are four choices for the permutation parameterization of the CKM matrix which we label according to the sign of θ_{12} and θ_{23}:

	θ_{12}	θ_{23}	θ_{13}	δ	θ_{123}
Standard	+0.22766	+0.04148	+0.00359	1.2023	
Perm ++	+0.22767	+0.04143	+0.00345		+0.00312
Perm +-	+0.22767	−0.04142	−0.00343		−0.00321
Perm -+	−0.22766	+0.04142	−0.00326		−0.00367
Perm −	−0.22766	−0.04140	+0.00320		+0.00378

(49)

The new parameters are small because the CKM matrix is near the unit matrix. In addition, the CKM matrix is very close to symmetric and this makes the θ_{123} parameter very small. Since it's not possible to distinguish the complex conjugate of the CKM matrix, there are four more choices; these have θ_{123} the same as above, while the θ_{jk} are negated. The unitary 1-magic CKM matrix is approximately:

$$\begin{bmatrix} +0.94835+0.22285i & +0.05280-0.21946i & -0.00115-0.00339i \\ +0.05877-0.21779i & +0.93820+0.25917i & +0.00302-0.04138i \\ -0.00713-0.00506i & +0.00900-0.03971i & +0.99813+0.04477i \end{bmatrix}. \quad (50)$$

for the "Perm ++" case where all parameters are positive.

From the mixing angles Eq. (49), it's clear that small changes to θjk for the standard parameterization correspond to equivalent small changes in the new parameterization. Near the CKM matrix, the δ angle changes approximately as:

$$\Delta\delta = 249\Delta\theta_{123} + 81\Delta\theta_{13} - 29\Delta\theta_{23} - 5.5\Delta\theta_{12}. \quad (51)$$

8 The MNS Mixing Matrix

In 2002, a simple form for the MNS lepton mixing matrix was proposed, the tribimaximal. [17] A tribimaximal unitary matrix has magnitudes as follows:

$$\begin{bmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{bmatrix}. \quad (52)$$

"Tribimaximal" refers to the center and right columns, which have three and two entries, respectively, with maximal mixing. The unitary version of this matrix transforms a vector of charged lepton amplitudes $(e, \mu, \tau)^\dagger$ to a vector of neutrino amplitudes $(\nu_1, \nu_2, \nu_3)^\dagger$.

Applying the decomposition Eq. (45) to the tribimaximal amplitudes, we find a tribimaximal unitary matrix:

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} + i \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{bmatrix}. \quad (53)$$

This form is symmetrically divided into a real part where the values $\{1/\sqrt{3}, 1/\sqrt{6}, 0\}$ are constant over the even permutations of generation, while the imaginary part is similar, but for the odd permutations. For example, $i/\sqrt{3}$ appears in the matrix elements that correspond to the permutation (23), the swap of the 2nd and 3rd generations.

9 Conclusion

By analyzing the unitary mixing matrices from the point of view of quantum information theory, we show that they appear to form a natural and elegant Lie subgroup of the unitary matrices. The resulting parameterization for 3×3 unitary matrices has four mixing angles; those that are equivalent under the permutation group are treated equivalently. This is more elegant than the arbitrary parameterizations now in use.

The need for a parameterization natural to the MNS matrix is apparent. For example, Bjorken, Harrison and Scott give a one parameter generalization of the tribimaximal matrix [18], Mohapatra and Yu give a two parameter generalization, [19], King gives a first-order approximation of a complete parameterization, [20] and Pakvasa, Rodejohann and Weiler give a parameterization around the tribimaximal form. [21] In this paper we provide a parameterization that has the triple advantages of treating the mixing angles equally, putting the tribimaximal matrix in a particularly symmetric form Eq. (53), and being related to a parameterization natural for the CKM matrix.

It's clear that the $MU(n)$ subalgebra does exist for all n; we've shown that it is equivalent to $U(n-1)$. What's still in question is whether any unitary matrix may be brought into $MU(n)$ form by multiplication of rows and columns by complex phases. This question appears to be related to an important unsolved problem in quantum information theory, but seems likely to be more tractable.

10 Acknowledgments

The author thanks Marni Sheppeard for pointing out the importance of 1-circulant and 2-circulant matrices in discrete Fourier transforms of matrices, and for pointing out that the CKM matrix is nearly the sum of a 1-circulant and a 2-circulant matrix; and Philip Gibbs for proving that the parameterization of 3×3 unitary matrices is complete.

References

1. C. S. Lam. Magic neutrino mass matrix and the Bjorken-Harrison-Scott parameterization. *Phys. Lett. B*, 640:260–262, 2006. hep-ph/0606220.
2. G. Mauro D'Ariano, Matteo G. A. Paris, and Massimiliano F. Sacchi. Quantum tomography. *Advances in Imaging and Electron Physics*, 128: 205–308, 2003. quant-ph/0302028.

3. I. D. Ivonovic. Geometrical description of quantal state determination. *J. Phys. A*, 14:3241–3245, 1981. abstract.
4. William K. Wootters and Brian D. Fields. Optimal state-determination by mutually unbiased measurements. *Ann. Phys.*, 191:363–381, 1989. abstract.
5. Julian Schwinger. The algebra of microscopic measurement. *Proc. N. A. S.*, 45:1542–1553, 1959. www.ncbi. nlm.nih.gov/pmc/articles/PMC222753/.
6. Julian Schwinger. The geometry of quantum states. *Proc. N. A. S.*, 46: 257–265, 1960. www.ncbi.nlm.nih.gov/pmc/articles/PMC222824/.
7. Thomas Durt, Berthold-Georg Englert, Ingemar Bengtsson, and Karol Życzkowski. On mutually unbiased bases. [quant-ph] 1004.3348.
8. Philip Gibbs. 3x3 unitary to magic matrix transformations, 2009. vixra 0907.0002.
9. Shivaramakrishnan Pancharatnam. Generalized theory of interference, and its applications. part I. coherent pencils. *Proc. Ind. Acad. Sci.*, pages 247–262, 1956.
10. Michael Berry. Quantal phase factors accompanying adiabatic changes. *Proc. R. Soc. Lond.*, A392:45–47, 1984. [abstract].
11. Rajendra Bhandari. On geometric phase from pure projections. *J.Mod.Opt.*, 45(10):2187–2195, 1998. physics/9810045.
12. Byron P. Roe. *Particle Physics at the New Millenium.* Springer-Verlag, 1996.
13. Cecilia Jarlskog. Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP nonconservation. *Phys. Rev. Lett.*, 55:1039–1042, 1985. [abstract].
14. Werner Rodejohann. Leptogenesis, mass hierarchies and low energy parameters. *Phys. Lett. B*, 542:100–110, 2002. hep-ph/0207053.
15. Carl Brannen. Spin path integrals and generations. Accepted at Foundations of Physics, 2010. [quant-ph] 1006.3114.
16. CKMFitter group. CKMfitter global fit results as of Beauty 09. www.ckmfitter.in2p3.fr, 2009. [link].
17. Paul F. Harrison, Don H. Perkins, and W. G. Scott. Tri-bimaximal mixing and the neutrino oscillation data. *Phys. Lett. B*, 530:167–173, 2002. hep-ph/0202074.
18. James D. Bjorken, Paul F. Harrison, and W. G. Scott. Simplified unitarity triangles for the lepton sector. *Phys. Rev. D*, 74:073012, 2006. hep-ph/0511201.
19. Rabindra N. Mohapatra and Hai-Bo Yu. Connecting leptogenesis to CP violation in neutrino mixings in a tri-bimaximal mixing model. *Phys. Lett. B*, 644:346–351, 2007. hep-ph/0610023.
20. Stephen F. King. Parametrizing the lepton mixing matrix in terms of deviations from tri-bimaximal mixing. *Phys. Lett. B*, 659:244–251, 2008. [hep-ph] 0710.0530.
21. Sandip Pakvasa, Werner Rodejohann, and Thomas J. Weiler. Triminimal parametrization of the neutrino mixing matrix. *Phys. Rev. Lett.*, 100: 111801, 2008. [hep-ex] 0711.0052.