Spin Path Integrals and Generations

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Abstract

Two consecutive measurements of the position of a non relativistic free particle will give entirely unrelated results. Recent quantum information research by G. Svetlichny, J. Tolar, and G. Chadzitaskos have explained this property of position observables as a result of a path in the Feynman integral being mathematically defined as a product of incompatible states; that is, as a product of mutually unbiased bases (MUBs). On the other hand, two consecutive measurements of a particle's spin give identical results. This raises the question "what happens when spin path integrals are computed over products of MUBs?" We show that the usual Pauli spin is obtained in the long time limit along with three generations of particles. We propose applications to the masses and mixing matrices of the elementary fermions.

1 Introduction

The position and momentum of a quantum particle are complementary observables. Complete knowledge of one implies no knowledge of the other. The transition probabilities between basis states of two complementary observables are all equal. Two bases with this property are called "mutually unbiased bases" or MUBs. For an m-dimensional Hilbert space, there can be at most $(m+1)$ bases that are pairwise mutually unbiased. A collection of such bases is called a "complete set of mutually unbiased bases".

For the case that m is the power of a prime it is known that complete sets of MUBs exist. The problem of finding a complete set of MUBs for general integers, or proving that such does or does not exist, is an unsolved problem in quantum information theory.

Since the Pauli spin matrices are defined in a 2-dimensional Hilbert space, a complete set of MUBs consists of three bases. Since the transition probabilities between spin-1/2 in perpendicular directions are always 1/2, a complete set of Pauli spin MUBs consists of three bases for spin in perpendicular directions. The canonical choice is spin in the $\vec{x}$, $\vec{y}$, and $\vec{z}$ directions.

Measuring the position of a free electron introduces uncertainty into its momentum. This uncertainty implies that a second measurement will be likely to find the electron far away. This behavior is the opposite of what occurs when the electron's spin is measured; consecutive measurements of the spin of an electron give the same result. We will attempt to reconcile these opposite behaviors by proposing that at the fundamental level, the electron's spin is also unstable; it takes on consecutive states from the MUBs of the Pauli spin matrices. Since a complete set of Pauli MUBs consists of three bases, we will call this new type of spin "tripled Pauli spin", and distinguish it from the traditional spin, "spin-1/2".

The kind of relationship between spin-1/2 and tripled Pauli spin that we are proposing is similar to the relationship between parity and position. Linear superposition allows a smooth state with even or odd parity to be assembled from position eigenstates. Parity of free particles is preserved even though their position is not. By looking at the long time propagators for tripled Pauli spin, we will find that spin-1/2 can be assembled from the linear superposition of tripled Pauli spin.

In attributing the activity of the electron's spin-1/2 to movement among mutually unbiased bases we are following the work of G. Svetlichny [1], and J. Tolar and G. Chadzitaskos [2] on the structure of the Feynman integral for the position observable over short time intervals. Feynman integrals consist of sums over paths. Each path is represented by a complex number. What Svetlichny noticed is that these complex numbers have magnitudes that do not depend on the path; only the phase depends on the path. This is the same behavior seen in the products of MUBs.

2 Tripled Pauli Spin

To measure spin-1/2 we first choose a unit vector $\vec{u}$ and then measure spin-1/2 with respect to that direction. The usual way of describing the results of the measurement is to say that we obtain either $+\hbar/2$ or $-\hbar/2$. But this terminology fails to include the direction along which spin-1/2 was measured. Instead we will say that the results of a spin-1/2 measurement in the u direction always has magnitude $\hbar/2$ and but has direction one of the two possible values $+\vec{u}$ or $-\vec{u}$.

Tripled Pauli spin has three times as many possible values as traditional spin-1/2. To measure tripled Pauli spin we first choose a set of perpendicular unit vectors, for example, $\vec{x}$, $\vec{y}$, and $\vec{z}$. The results of the measurement will always have the same magnitude $\hbar/2$, but the direction will be one of the six possible vectors $\pm\vec{x}$, $\pm\vec{y}$, or $\pm\vec{z}$. A classical measurement of spin would give the magnitude and an arbitrary direction; tripled Pauli spin is intermediate between the traditional classical and quantum measurements.

When the Pauli algebra is represented by the usual Pauli spin matrices, a choice of basis states for spin-1/2 in the $\pm z$ direction is:

$$|+\vec{z}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\vec{z}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

Note that these basis states are not uniquely defined; they can be multiplied by arbitrary complex phases to give an equally valid basis. The above is the traditional choice. A choice for the basis states for spin-1/2 in the $\pm x$ direction is:

$$|+\vec{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\vec{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 \\ -1 \end{pmatrix}. \tag{2}$$

The transition probabilities between the states of these two bases are all equal to 1/2:

$$|\langle\pm\vec{x}|\pm\vec{z}\rangle|^2 = 1/2. \tag{3}$$

Thus these two bases, $\{|+\vec{z}\rangle, |-\vec{z}\rangle\}$ and $\{|+\vec{x}\rangle, |-\vec{x}\rangle\}$ are mutually unbiased, and similarly for $\{|+\vec{y}\rangle, |-\vec{y}\rangle\}$.

Since the Pauli spin matrices are written with only σ_z diagonalized, only the basis states $|+\vec{z}\rangle$ and $|-\vec{z}\rangle$ are pretty. Our work will be simpler if we use a notation that eliminates the arbitrary complex phases of the spinor basis states; we will work with pure density matrices.

Given a ket $|a\rangle$, the associated pure density matrix is obtained by multiplying by the bra: $|a\rangle\langle a|$. This cancels the arbitrary complex phase of the ket. Converting the $\{|\pm\vec{z}\rangle\}$ spinor basis set into pure density matrices we find the following two states:

$$|\pm\vec{z}\rangle \rightarrow |\pm\vec{z}\rangle\langle\pm\vec{z}| = (1 \pm \sigma_z)/2. \tag{4}$$

This conversion simplifies the representation of the $\pm\vec{x}$ and $\pm\vec{y}$ states; their pure density matrices are $(1 \pm \sigma_x)/2$ and $(1 \pm \sigma_y)/2$. For any unit vector $\vec{u}$, we can write the pure density matrices corresponding to spin-1/2 in the $\vec{u}$ direction as $(1 \pm \sigma_u)/2$ where $\sigma_u = u_x\sigma_x + u_y\sigma_y + u_z\sigma_z$.

When one converts a basis set into pure density matrix form, the features which characterize a basis set are transformed into algebraic relations among the pure density matrices. The normality of the kets become idempotency:

$$\langle\pm\vec{u}|\pm\vec{u}\rangle = 1 \rightarrow [(1 \pm \sigma_u)/2]\,[(1 \pm \sigma_u)/2] = (1 \pm \sigma_u)/2, \tag{5}$$

and orthogonality becomes annihilation:

$$\langle+\vec{u}|-\vec{u}\rangle = 0 \rightarrow [(1 + \sigma_u)/2]\,[(1 - \sigma_u)/2] = 0. \tag{6}$$

Finally, the requirement that the number of basis elements equals the dimensionality of the vector space becomes the requirement that the sum of the basis elements is unity:

$$(1 + \sigma_u)/2 + (1 - \sigma_u)/2 = 1. \tag{7}$$

These properties generalize to higher dimension. We will find the same relations among the long term propagators of tripled Pauli spin. We will interpret the three different long term propagators as three generations of elementary spin-1/2 fermions.

3 Pauli MUB Arithmetic

The wave function ψ at a spacetime point x' can be determined from the wave function at other spacetime points x by integration with a Green's function:

$$\psi_{out}(x') = \int G(x',x)\psi_{in}(x)\;dx. \tag{8}$$

Since our concern is with the spin of a free electron, and not its momentum or energy, the integral becomes a simple matrix multiplication. An intermediate simplification is to keep track of energy but to ignore momentum. This simplification of Feynman path integrals is described in a paper by Bialynicki-Birula and Sowiński [3] on the quantum electrodynamics of qubits.

Under the usual spin-1/2, paths through spin space have two possible states at each vertex, say spin-1/2 in the $+\vec{v}$ and $-\vec{v}$ directions. The spin propagators between these vertices are the projection operators for spin-up and spin-down. The vertices consist of unit matrices. For a free electron there is no way for spin-1/2 to change so we can restrict our attention to the $+\vec{v}$ case. Then there is only one propagator, $(1+\sigma_v)/2$. This propagator takes states of spin-1/2 $= +\vec{v}$ and leaves them unchanged.

In converting from spin-1/2 to tripled Pauli spin, the single state $(1+\sigma_v)/2$ is expanded into three states, $(1+\sigma_x)/2$, $(1+\sigma_y)/2$, and $(1+\sigma_z)/2$. When we use these states for tripled Pauli spin, to reduce confusion we will call them "forms" rather than "states". We abbreviate them as:

$$\begin{aligned} X &= (1+\sigma_x)/2, \\ Y &= (1+\sigma_y)/2, \\ Z &= (1+\sigma_z)/2. \end{aligned} \tag{9}$$

When we need the spin-down states, we will label them with a bar so that $\bar{X} = (1-\sigma_x)/2$, etc.

Each vertex will again be the unit matrix, but this time there will be interactions between propagators of different forms. For example, there are three possible paths of three forms, that begin and end with tripled Pauli spin $+z$:

$$\begin{aligned} Z\;X\;Z &= \sqrt{1/2^2}Z, \\ Z\;Y\;Z &= \sqrt{1/2^2}Z, \\ Z\;Z\;Z &= \sqrt{1/2^0}Z. \end{aligned} \tag{10}$$

The three products are all real multiples of Z. If X, Y and Z are thought of as pure density matrices, the real factors in the above calculations are to transition amplitudes. With ZXZ and ZYZ there are two transitions, each with amplitude $\sqrt{1/2}$, and with $ZZZ = Z$ there are no transitions so the amplitude is 1.

More general products can be complex:

$$Z\;X\;Y\;Z = \sqrt{+i/2^3}Z, \tag{11}$$

where we define $\sqrt{\pm i} = \exp(\pm i\pi/4)$. Here there are three transitions between MUB basis states so the magnitude is $\sqrt{1/2^3}$. A path that makes N transitions through different forms will end up with a probability of 2^{-N} and therefore its amplitude will have a magnitude of $\sqrt{2^{-N}}$.

The complex phase $\sqrt{i}$ in Eq. (11) is a geometric phase, also called Berry [4] or Pancharatnam [5] phase. These phases can be picked up when a quantum particle goes through a series of states and returns to its initial state. It does not depend on the arbitrary complex phases of spinors and consequently is an observable. Our use of quantum phase will arise from products of projection operators of the sort described in R. Bhandari's paper [6], but for spin-1/2 rather than photon polarization.

In computing sums of MUB Feynman paths, we will use X, Y, and Z as part of a basis for a complex vector space. These three account for paths that begin and end with the same form. Path integrals whose final forms are different from their initial forms must be handled differently. For these mixed paths there are six ways to choose different initial and final forms.

There are three possible paths of three forms that begin with Z and end with X:

$$\begin{array}{rcl} X\;X\;Z &=& \sqrt{1/2^0}\,XZ, \\ X\;Y\;Z &=& \sqrt{+i/2^1}\,XZ, \\ X\;Z\;Z &=& \sqrt{1/2^0}\,XZ. \end{array} \tag{12}$$

In the above calculation, the three products are all complex multiples of the same matrix: XZ. Twice this matrix is idempotent and has trace 1:

$$(2XZ)^2 = 2XZ = \left(\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right). \tag{13}$$

This is a property of all MUB paths that begin and end with different forms; they will be complex multiples of an idempotent, with X and Z replaced by appropriate forms. Products that begin and end with different forms from the same basis are raising and lowering operators. For example $ZX\bar{Z}$ is a raising operator for spin-1/2.

The pure density matrices are usually defined as the matrices that can be produced from normalized spinors. An alternative definition is that they are the Hermitian matrices that are idempotent and have trace 1. A spinor corresponding to a pure density matrix can be obtained by taking any nonzero column of the matrix, treating the column as a vector, and normalizing it. The matrix used in Eq. (13), and the other five obtained by replacing X and Z with different forms, possesses two of the three properties that define a pure density matrix; they are idempotent and have trace 1 but they are not Hermitian.

Hermiticity is associated with the property of time reversal invariance. We have:

$$(2XZ)^\dagger = 2ZX. \tag{14}$$

More generally, Hermitian conjugation reverses the order of a product of any number of forms.

For convenience, we will drop off the factor of 2 and use

$$X, \quad Y, \quad Z, \quad XY, \quad YX, \quad XZ, \quad ZX, \quad YZ, \quad ZY \tag{15}$$

as the basis for MUB spin path integrals. Any product of MUB forms can be written as a complex multiple of one of these nine. In using these, we have five more than are needed to give a basis for 2×2 complex matrices, but what we are looking for is a basis for paths and path integral calculations.

In the usual Feynman integrals, two paths are added (integrated) together only if they begin with the same state and also end with the same state. Thus one specifies initial and final states of the particles and sums over diagrams that relate those states. For the MUB case we will also be adding two paths together only if they begin and end with the same states. For this reason, our set of nine products of forms can be used as if they were a basis set for a complex vector space. A vector in that space is a collection of nine amplitudes. Adding two such vectors together is equivalent to adding nine Feynman integrals to nine other Feynman integrals to get nine sums. In this way, given a path, (or a collection of MUB paths that all begin with the same form and end with the same form), we associate a complex number. Our notation keeps track of the initial and final forms.

In addition to summing two paths, we also need to connect one path to another. This is similar to the multiplication of two products of forms: Suppose we have two paths (or sums of paths) A and B which are associated with complex numbers a and b. If it so happens that path A ends with the same form that path B begins with, then the concatenation of these paths, call it BA, will be associated with the complex number $ba = ab$ multiplied by non commutative corrections. This suggests that we should organize our calculations so that transitions between forms are kept inside the path integral basis Eq. (15); we can then use complex multiplication to model the concatenation of paths.

Accordingly, we will only concatenate two path integral basis elements if the final form of the first element matches the initial form of the second element. For example, this requirement allows $(XY)(YZ)$ or $(Z)(ZX)$ but not $(XY)(Z)$ or $(Z)(X)$. Since the path integral basis has nine elements, there are a total of 81 possible products of them but the restriction reduces the number of products we will consider to 27. Using idempotency, any spin path can be rewritten to satisfy requirement, see Eq. (25) below.

Of the 27 products of path integral basis elements we will consider, 15 are already correct for standard complex multiplication. The five that begin with X are:

$$\begin{aligned} (X)(X) &= X, \\ (X)(XY) &= XY, \\ (X)(XZ) &= XZ, \\ (XY)(Y) &= XY, \\ (XZ)(Z) &= XZ. \end{aligned} \tag{16}$$

The other ten such products are obtained by cyclic permutation of X, Y, and Z. We will call these the "diagonal products" for reasons which will be clear below. The remaining 12, off diagonal products are more complicated. The four that begin with X are:

$$\begin{array}{rcl} (XY)(YX) &=& \sqrt{1/2^2}\,X, \\ (XZ)(ZX) &=& \sqrt{1/2^2}\,X, \\ (XY)(YZ) &=& \sqrt{+i/2^1}\,XZ, \\ (XZ)(ZY) &=& \sqrt{-i/2^1}\,XY. \end{array} \tag{17}$$

The other eight are obtained by cyclic permutation.

Other than the complex factors of the off diagonal products, the above 27 products are compatible with matrix multiplication. Let $(a'_x, a'_y, ...a'_{yz})$ be a 9-vector of complex numbers associated with a collection of tripled Pauli spin paths. We use the prime to indicate that these represent complex multiples of non commutative matrices. For the moment we will suppose that there is some single path associated, perhaps of type XZ, so that only one of them (i.e. a'_{xz}) is nonzero. Assemble them into a 3×3 matrix a' as follows:

$$a' = \left(\begin{array}{ccc} a'_x & a_{xy} & a'_{xz} \\ a'_{yx} & a'_y & a'_{yz} \\ a'_{zx} & a'_{zy} & a'_z \end{array} \right) \tag{18}$$

Let b' be a similar matrix for another path whose final form matches the initial form of a'. For instance, b' could represent a path of type ZY so that only b'_{zy} is nonzero. Concatenating the paths gives a path of type XY. Corresponding to this, the matrix product ab' will have only one nonzero entry, $ab'_{xy} = a'_{xz}b'_{zy}$. This is not quite correct; according to Eq. (17) we should have $ab'_{xy} = \sqrt{-i/2^1}a'_{xz}b'_{zy}$.

To fix the matrix product, we need to scale the off diagonal elements (i.e. a'_{xy}, a'_{yz}, a'_{zx}, a'_{yx}, a'_{zy}, a'_{xz} and similarly for b') in such a way that the off diagonal products are corrected without changing the diagonal products. Rewriting Eq. (17) in terms of what it says about products of the elements of a' and b' (and still assuming that only one element of a' and b' are nonzero) what we want are new matrices, a, b, and ab such that:

$$\begin{array}{rcl} \sqrt{1/2^2}\,ab_x &=& a_{xy}b_{yx}, \\ \sqrt{1/2^2}\,ab_x &=& a_{xz}b_{zx}, \\ \sqrt{+i/2^1}\,ab_{xz} &=& a_{xy}b_{yz}, \\ \sqrt{-i/2^1}\,ab_{xy} &=& a_{xz}b_{zy}, \end{array} \tag{19}$$

while leaving $ab_x = a_xb_x$ and the same cyclically for y and z. A suitable transformation is $a' \to a$ by:

$$\begin{array}{rcl} a_x &=& a'_x, \\ a_{xy} &=& \eta_g\; a'_{xy}, \\ a_{xz} &=& \eta_g^*\; a'_{xz}, \end{array} \tag{20}$$

where

$$\eta_g = \sqrt{1/2}\,e^{+i\pi/12}e^{2ig\pi/3}, \;\text{ for } g = 1, 2, 3, \tag{21}$$

and cyclic permutations give the transformations on the other six path basis elements. The complex phase $2i\pi/3$ appears repeatedly so we will abbreviate it as w:

$$w = \exp(2i\pi/3). \tag{22}$$

So $\eta_g = \sqrt{1/2}\ \exp(i\pi/12)\ w^g$. We will use the integer parameter g to represent the generation quantum number.

Since a tripled Pauli spin state is a complex 3-vector, a general propagator for tripled Pauli spin will be represented by a 3×3 complex matrix. The matrix entries represent the nine cases: tripled Pauli spin changing from $+x$ to $+x$, $+x$ to $+y$, etc. Suppose this propagator is followed by another propagator b. In computing the propagator ab we must sum over all possible paths. With matrices a and b, this simply amounts to matrix multiplication. For example, the three terms on the right side of

$$ab_{xz} = a_x b_{xz} + a_{xy} b_{yz} + a_{xz} b_z. \tag{23}$$

correspond to the three paths that go through the new node with tripled Pauli spin $+x$, $+y$, and $+z$, respectively. Thus we have transformed the problem of concatenating the propagators of tripled Pauli spin into 3×3 matrix multiplication.

4 Tripled Pauli Spin Propagator

Let's begin by computing the propagator for tripled Pauli spin $+\vec{x}$ to $+\vec{z}$ with two internal paths, call it $G_2(+\vec{z}, +\vec{x})$. This is a sum over paths with four forms. The initial form is X, the final form Z while the two inner forms can be any of X, Y, or Z. Taking all possible cases for the inner forms and calculating with the Pauli spin matrices we find:

$$\begin{aligned} G_2(+\vec{z}, +\vec{x}) &= ZXXX + ZXYX + ZXZX \\ &+ ZYXX + ZYYX + ZYZX \\ &+ ZZXX + ZZYX + ZZZX, \\ &= (3 + 3i/4)ZX. \end{aligned} \tag{24}$$

This is clearly too large; the outgoing amplitude is larger in magnitude than the incoming and we haven't yet included the other eight paths from $\{X, Y, Z\}$ to $\{X, Y, Z\}$.

The problem is that the transition amplitudes need to be adjusted for the fact that we have added new possible paths at each vertex. To preserve probability at each vertex, we need to make the transition probabilities smaller. There are three vertices so we will multiply by a factor κ^3, one κ for each vertex.

We will compute κ later in this section, for now, let's see how to rewrite path concatenation as a matrix multiplication. We are concerned with paths that look like $(Z)(P)(Q)(X)$ where (P) and (Q) are the inner forms. First, using idempotency, we duplicate the inner forms and refactor into pairs:

$$\begin{aligned} ZPQX &= Z(PP)(QQ)X, \\ &= (ZP)(PQ)(QX). \end{aligned} \tag{25}$$

Modify the pairs to replace $(XX) = (X)$, $(YY) = (Y)$, and $(ZZ) = (Z)$. Now we have the path as a product of our nine path basis elements in such a way that each adjacent path integral basis elements match their adjacent projection operators.

The state ZP has three possible choices for P so represent it as κ times the matrix with 1s in the bottom (z) row:

$$zp' = \kappa \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{26}$$

Convert the matrix zp' to zp to obtain:

$$zp = \kappa \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta_g^* & \eta_g & 1 \end{pmatrix} \tag{27}$$

Similarly, there are nine possible choices for the middle term (PQ) so it will be a matrix with all elements equal to κ. This is the general propagator for a single MUB step. Transforming it to pq we have:

$$pq = \kappa \begin{pmatrix} 1 & \eta_g^* & \eta_g \\ \eta_g & 1 & \eta_g^* \\ \eta_g^* & \eta_g & 1 \end{pmatrix}. \tag{28}$$

Finally, (QX) will be κ times a matrix whose left (x) column only is nonzero, with all entries 1. It converts to:

$$qx = \kappa \begin{pmatrix} 1 & 0 & 0 \\ \eta_g & 0 & 0 \\ \eta_g^* & 0 & 0 \end{pmatrix}, \tag{29}$$

and the sum over paths is represented by the matrix product:

$$\kappa^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta_g^* & \eta_g & 1 \end{pmatrix} \begin{pmatrix} 1 & \eta_g^* & \eta_g \\ \eta_g & 1 & \eta_g^* \\ \eta_g^* & \eta_g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \eta_g & 0 & 0 \\ \eta_g^* & 0 & 0 \end{pmatrix}. \tag{30}$$

Upon inserting the Pauli spin matrices we obtain:

$$\begin{aligned} G_2(+\vec{z}, +\vec{x}) &= 3\kappa^3[(\eta_g + (\eta_g *)^2 + \eta_g^* \eta_g^2)/\eta_g] \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ &= 3\kappa^3[(\eta_g + (\eta_g *)^2 + \eta_g^* \eta_g^2)/\eta_g] \; ZX \end{aligned} \tag{31}$$

Replacing η_g with $\sqrt{1/2}\exp(i\pi/12)w^g$, we find that the result does not depend on g and is the same as the sum over paths, Eq. (24).

The above result generalizes. Longer path integrals introduce extra factors of the center array of Eq. (30) which we will call G_g. This will replace G_g with powers of G_g:

$$G_g^N = \begin{pmatrix} 1 & \eta_g & \eta_g^* \\ \eta_g^* & 1 & \eta_g \\ \eta_g & \eta_g^* & 1 \end{pmatrix}^N \tag{32}$$

Note that the above G_g matrix is 1-circulant, that is each row is identical to the row above, but rotated 1 position to the right.

The discrete Fourier transform diagonalizes 1-circulant matrices. This allows us to compute G_g^N by taking the discrete Fourier transform, taking powers of the diagonal entries, and then reverse transforming. Thus we can solve for G_g^N in closed form.

Define the Fourier transform matrix F as

$$F = \frac{1}{\sqrt{3}} \begin{pmatrix} w & w^* & 1 \\ w^* & w & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{33}$$

so that the discrete Fourier transform of a vector $\vec{v}$ is $F\vec{v}$. Then the discrete Fourier transform of a matrix M is

$$\tilde{M} = F\ M\ F^*. \tag{34}$$

The discrete Fourier transform takes a 1-circulant matrix with top row (A, B, C) to a diagonal matrix with the discrete Fourier transform of the vector (A, B, C) down the diagonal:

$$\begin{pmatrix} A + w^*B + wC & 0 & 0 \\ 0 & A + wB + w^*C & 0 \\ 0 & 0 & A + B + C \end{pmatrix} \tag{35}$$

The transform of G_g is a diagonal matrix $\tilde{G}_g$ of this form. The jth element on the diagonal of G_g is:

$$\kappa[1 + \sqrt{2}\cos(2g\pi/3 + 2j\pi/3 - \pi/12)]. \tag{36}$$

Note that the G_g have the same diagonal elements but in an order that depends on g and that $G_1 + G_2 + G_3 = 1$. We have a closed form solution for G_g^N:

$$G_g^N = F^*(\tilde{G}_g)^N\ F; \tag{37}$$

since $\tilde{G}_g$ is diagonal, the above is simple to compute.

The three diagonal entries of $\tilde{G}_g$ are multiplied by κ. In the limit as $N \to \infty$, the largest diagonal entry of $(\tilde{G}_g)^N$ will dominate. In order for this limit to exist, that diagonal entry must be unity. Therefore we have that

$$\kappa = 1/(1 + \sqrt{2}\cos(\pi/12)) = 3/(3/2 + \sqrt{3}/2). \tag{38}$$

One over κ gives the amount by which we were violating preservation of probability before including κ.

The other nonzero entry in $\tilde{G}$ is $(1+\sqrt{2}\cos(7\pi/12))/\kappa = 2-\sqrt{3}$. Thus the explicit formula for G^N is

$$G^N = F^* \begin{pmatrix} (2-\sqrt{3})^N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} F \tag{39}$$

where F is given in Eq. (33). Finally, to obtain the non commutative amplitudes, one performs the reverse transformation.

In the limit as $N \to \infty$, the other entry on the diagonal of G^N goes to zero and we have:

$$G^\infty = F^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} F = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{40}$$

Thus, in the limit, all nine of the transition probabilities are equal to $(1/3)^2 = 1/9$. On converting the above to non commutative form, we find three different long term propagators, depending on the generation g:

$$G_g^{'\infty} = \frac{1}{3\sqrt{2}} \begin{pmatrix} \sqrt{2} & e^{-i\pi/12}w^{-g} & e^{+i\pi/12}w^{+g} \\ e^{+i\pi/12}w^{+g} & \sqrt{2} & e^{-i\pi/12}w^{-g} \\ e^{-i\pi/12}w^{-g} & e^{+i\pi/12}w^{+g} & \sqrt{2} \end{pmatrix}. \tag{41}$$

The above can be translated from the matrix using the path basis to obtain:

$$\begin{aligned} G_g^{'\infty} \;=\;& [(X+Y+Z) + e^{-i\pi/12}w^{-g}\sqrt{1/2}(XY+YZ+ZX) \\ & +e^{+i\pi/12}w^{+g}\sqrt{1/2}(YX+ZY+XZ)]/3. \end{aligned} \tag{42}$$

We will associate these three propagators with the three generations of elementary fermions.

5 The Particle Generations

The long term propagators $G_g^{'\infty}$ are idempotent:

$$(G_g^{'\infty})^2 = G_g^{'\infty}, \tag{43}$$

they annihilate each other:

$$G_g^{'\infty} G_h^{'\infty} = 0, \;\; \text{if } g \neq h, \tag{44}$$

and their traces are 1. These are just the requirements needed to show that a matrix is a pure density matrix and so these are associated with pure states. The analogous statement about the spin-1/2 propagators is that $(1+\sigma_v)/2$ and

$(1-\sigma_v)/2$ are idempotent, annihilate each other, and have trace 1 and so are pure density matrices.

The off diagonal elements of Eq. (41) are each multiplied by a factor of w^{+g} or w^{-g}. Since the sum of powers of w is zero, i.e,

$$w + w^2 + w^3 = w^{-1} + w^{-2} + w^{-3} = 0, \tag{45}$$

the three matrices $G_g^{'\infty}$ sum to the unit matrix:

$$G_1^{'\infty} + G_2^{'\infty} + G_3^{'\infty} = 1. \tag{46}$$

This shows that their spinor versions form something similar to a complete basis for the 3×3 matrices (i.e. a complete basis for the path integrals of the tripled Pauli spin products of X, Y, and Z). More accurately, these $G^{'\infty}$ are 3×3 matrices of 2×2 matrices so $\{G_g^{'\infty}\}$ gives half the basis for 6-vectors. The other half of the basis is obtained from the tripled Pauli spin long term propagators for spin-1/2 $-\vec{v}$.

Analogously, the spin-1/2 propagators $(1 \pm \sigma_z)/2$ sum to unity and so the associated states form a basis for the 2×2 matrices. Thus the assumption of tripled Pauli spin leads to a splitting of the spin-1/2 states spin-up and spin-down into three forms each. This gives us three generations of spin-1/2 particles.

The factors w^{+g} and w^{-g} that distinguish the three generations are cubed roots of unity. Since the long term propagators are otherwise identical, this suggests that the generations should be simple when characterized as cube roots of unity. That is, w^{+g} are the three solutions to the equation:

$$z^3 = 1. \tag{47}$$

Better, if we normalize the diagonal of Eq. (41) to unity, the off diagonal elements are $w^{\pm g} e^{i\pi/12}/\sqrt{2}$. These are the roots of the equation:

$$z^3 = (1+i)/4. \tag{48}$$

Consequently, we expect that the differences between the generations of elementary particles should be simple when expressed as functions of the complex roots of Eq. (48). Those roots are:

$$u_g = \exp(2ig\pi/3 + i\pi/12)/\sqrt{2}. \tag{49}$$

In the next subsections we will apply this theory to the fermion masses and mixing angles. This amounts to complexifying generation; we will extend generation from a discrete variable that takes on the three values 1, 2, and 3, to a complex variable that can take on any complex value. This extension of generation amounts to consideration of particle states that are not long term propagators of tripled Pauli spin.

5.1 Lepton Masses

In the standard model, mass is an interaction between the left and right handed spin-1/2 states. The mass terms in the Lagrangian, for the three generations, are:

$$\Sigma_{g=1}^{3}(m_g \psi_{gL}^{*}\psi_{gR} + m_g \psi_{gR}^{*}\psi_{gL}). \tag{50}$$

where m_g is the mass of the gth generation particle. When the ψ_{gL} and ψ_{gR} are rewritten in terms of tripled Pauli spin forms, it's natural to expect that the three forms of ψ_{gL} will transform into the three forms of ψ_{gR} and that generational differences in mass will be due to differences in how the forms interact. This provides hope that the three experimentally determined constants, m_g can be united into a single constant. The differences between m_g will then be determined by differences in the corresponding tripled Pauli spin propagators.

The various portions of the long term propagators differ only in that they depend on w^g. Consequently, under the assumption of tripled Pauli spin, it may be useful to write m_g in the form:

$$m_g = f(w^g) = \Sigma_{n=-\infty}^{+\infty} A_n (w^g)^n. \tag{51}$$

Since $w^3 = 1$, the infinite sum is reduced to:

$$m_g = \Sigma_{n=1}^{3} A_n w^{ng}. \tag{52}$$

This is $\sqrt{3}$ times the discrete Fourier transform of the 3-vector (A_1, A_2, A_3) so we can find A_n by the inverse discrete Fourier transform. Since the masses are real, we have that $A_1^* = A_2$ and A_3 is real. Putting $A_1 = B\exp(iC)$ we have:

$$\begin{aligned} m_g &= A_3 + A_1 \exp(2ig\pi/3) + A_2 \exp(-2ig\pi/3), \\ &= A_3 + 2B\cos(2g\pi/3 + C). \end{aligned} \tag{53}$$

This is only a slight variation of a charged lepton mass equation used by Gerald Rosen:

$$\sqrt{m_g} = 17.716 \text{ MeV } [1 + \sqrt{2}\cos(2g\pi/3 + 2/9)], \tag{54}$$

who notes that it is accurate to $O(10^{-5})$ and attributes it to a Dirac-Goldhaber model of the quarks and leptons.[7] In this formula, $C = 2/9$ and $B = A_3\sqrt{2}$. The above is similar to the long term propagators in that it includes a square root of 2, but different in that the angle $\pi/12$ is missing and instead there is an angle 2/9.

In 1982 [8, 9], Yoshio Koide discovered a formula for the charged lepton masses:

$$2(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 3(m_e + m_\mu + m_\tau). \tag{55}$$

Since then, measurements of the τ mass have converged to Koide's prediction; it is still accurate to well within experimental error. This formula follows from the $\sqrt{2}$ in Eq. (54).

The neutrino masses are known only by the differences in the squares of their masses implied by neutrino oscillation measurements. Koide's charged lepton mass equation was expanded to the neutrinos by the present author in 2006.[10, 11, 12, 13, 14] The method was to allow one of the square roots of the neutrino masses to be negative; without that the neutrino mass information is incompatible with Koide's mass formula. Koide's formula then provides a third restriction on the neutrino masses.

The resulting equation for the neutrino masses, in a form similar to that of Eq. (54) [10]:

$$\sqrt{m_{\nu g}} = 0.1000(26)\sqrt{\mathrm{eV}}\ [1+\sqrt{2}\cos(2g\pi/3+\pi/12+2/9]. \tag{56}$$

The 2006 predictions [10] for the neutrino masses, in electron volts, were:

$$\begin{aligned} m_{\nu 1} &= 0.00038, \\ m_{\nu 2} &= 0.0089, \\ m_{\nu 3} &= 0.0507, \end{aligned} \tag{57}$$

which give differences in the squares of the masses as:

$$\begin{aligned} \nabla m^2_{\mathrm{sol}} &= |m_1^2 - m_2^2| = 7.9\times 10^{-5}\ \mathrm{eV}^2, \\ \nabla m^2_{\mathrm{atm}} &= |m_2^2 - m_3^2| = 2.5\times 10^{-3}\ \mathrm{eV}^2. \end{aligned} \tag{58}$$

A recent measurements of the solar mass parameter $\nabla m^2_{\mathrm{sol}}$ from the Sudbury Neutrino Observatory [15] is $7.59(21)\times 10^{-5}$ eV^2. Data from MINOS [16] give $\nabla m^2_{\mathrm{atm}}$ around 2.43×10^{-3} eV^2, so the above mass predictions are still well within the error bars from three years ago. Adjusting the mass scale 0.1000(26) a little to match the new experimental data, a more current neutrino mass formula is:

$$\sqrt{m_{\nu g}} = 0.0990\ \sqrt{\mathrm{eV}}\ [1+\sqrt{2}\cos(2g\pi/3+\pi/12+2/9)], \tag{59}$$

which gives 7.7×10^{-5} and 2.44×10^{-3} eV^2.

The two lepton mass formulas, Eq. (54) and Eq. (59), are similar except for the mass scale and the angle. The two formulas have the common angle 2/9, a number which is close to the Cabibbo angle. They differ in that the neutrino takes the angle $\pi/12$ that appears in the tripled Pauli spin calculations. This suggests that there is some difference in the mass interaction between the two particles. Perhaps the neutrino mass interaction is a simple, low energy interaction while the charged lepton mass interaction is more complicated. It may be useful to note that massless spin-1 particles have the same quantum phase as spin-1/2 particles.

In analyzing the generations we claimed that the attributes of the generations should be simple when written in terms of the roots of Eq. (48). The obvious way to obtain the charged lepton masses in terms of u_g is:

$$\sqrt{m_g} = 17.716\ \mathrm{MeV}\ [1 + u_g e^{2i/9 - i\pi/12} + u_g^* e^{-2i/9+i\pi/12}], \tag{60}$$

the corresponding equation for the neutrinos is:

$$\sqrt{m_{\nu g}} = 0.0990\ \sqrt{\mathrm{eV}}\ [1 + u_g e^{2i/9} + u_g^* e^{-2i/9}], \tag{61}$$

The numerical constants in the above two equations, 17.716 and 0.0990, have a ratio very close to an exact power of three. That is, $17.716/0.0990 = 3^{11.009}$. We speculate that a theory explaining the mass hierarchy between the charged and neutral leptons will involve a coupling constant that is a power of 3 (such as the 1/3 of Eq. (40)), and that differences in the complexity of the mass interaction explains the mass hierarchy between the charged and neutral leptons.

The square roots in the mass formulas would arise naturally if the mass operator were written as the square of another operator. Energy operators frequently have squares in them such as the potential energy of the harmonic oscillator, x^2, or the energy in the electromagnetic field, $|\vec{E}|^2 + |\vec{B}|^2$.

5.2 Quark Masses

Since the quarks carry three color states, it's natural to speculate that the quarks are composed of the same tripled Pauli spin forms as the leptons, but mixed. The quarks with electric charge $+2/3$ would be built from two charged lepton anti-forms and one neutrino form, while the $-1/3$ quarks arise from one charged lepton form and two neutrino anti-forms. With such a model, the quarks will have residual tripled Pauli spin attributes due to their mixed character.

Since the quarks are forever bound in particles, their masses cannot be measured without resort to theoretical models. Different models give different quark masses. On the other hand, some of the hadron masses are well measured so it might be possible to fit them to analogies to the charged and neutral lepton mass formulas.

Regge trajectories [17] are mass relations that relate hadrons with different angular momentum. They were obtained by T. Regge from considering orbital angular momentum as a complex variable [18]. The present paper extends generation to a complex variable. Since the leptons all share the same angular momentum it is natural to expect that any applications to the hadrons will appear in pairs of triplets of hadrons with the same angular momentum quantum numbers. In the current view these would be radial excitations but only a few of the very heaviest lowest energy hadrons have been successfully modeled this way.

The path integral methods used in this paper ignores position. This approximation will fail badly when the quark's wave function has strong dependence on position. Thus our approximation is most accurate when applied to the very light quarks at high energy. This is the opposite of the usual non relativistic analysis of the quarks which is best for heavy quarks at low energy. Unfortunately, the hadrons of the very light quarks also have large relative uncertainties in energy and consequently their masses are difficult to measure accurately. In addition, we are most interested hadrons which have been carefully explored. These requirements make the heavy q-qbar mesons the natural place to look for analogies to the leptons [19].

Since there are six leptons split into two triplets, we expect the radial excitations to also show up in groups of six. Perhaps not coincidentally, there are known to be six states in each of the heavy, well explored, J/ψ (c-cbar) and Υ (b-bbar) mesons.

The six J/ψ can be fit with the following formulas:

$$\begin{array}{rcl}\sqrt{m_{\psi 0g}} &=& \mu_e[3.45417 - 0.17679(u_g e^{2i/9} + u_g^* e^{-2i/9})], \\ \sqrt{m_{\psi 1g}} &=& \mu_e[3.55037 - 0.06324(u_g e^{2i/9 - i\pi/12} + u_g^* e^{-2i/9 + i\pi/12})],\end{array} \quad (62)$$

where $\mu_e = 17.716\sqrt{\text{MeV}}$ is from the charged lepton mass equation. The $m_{\psi 0g}$ are the three lower mass states. These correspond to the neutrinos. The $m_{\psi 1g}$ correspond to the charged leptons. Note that the lighter triplet uses the neutral lepton formula while the heavier triplet uses the charged lepton formula. The above equations fit the six masses with the four fitted constants 3.45417, 0.17679, 3.55037, and 0.06324, removing two degrees of freedom. The resulting fits:

g	Meson	m_g	Exp.
01	$J/\psi(1S)$	3096.916	3096.916(.011)
02	$\psi(3770)$	3775.154	3775.2(1.7)
03	$\psi(4415)$	4421.063	4421.0(4.0)
11	$\psi(2S)$	3686.083	3686.093(.034)
12	$\psi(4040)$	4040.356	4039.0(1.0)
13	$\psi(4160)$	4149.827	4153.0(3.0)

(63)

are quite accurate.

The formulas for the Υ masses are:

$$\begin{array}{rcl}\sqrt{m_{\Upsilon 0g}} &=& \mu_e[5.64741 - 0.08957(u_g e^{2i/9} + u_g^* e^{-2i/9})], \\ \sqrt{m_{\Upsilon 1g}} &=& \mu_e[5.85093 - 0.05483(u_g e^{2i/9 - i\pi/12} + u_g^* e^{-2i/9 + i\pi/12})],\end{array} \quad (64)$$

The resulting fits:

g	Meson	m_g	Exp.
01	$\Upsilon(1S)$	9456	9460.30(.26)
02	$\Upsilon(2S)$	10035	10023.26(.31)
03	$\Upsilon(4S)$	10554	10579.4(1.2)
11	$\Upsilon(3S)$	10355.2	10355.2(.5)
12	$\Upsilon(10860)$	10864.4	10865.0(8.0)
13	$\Upsilon(11020)$	11019.5	11019.0(8.0)

(65)

are excellent for the three heavier Υ states but less than perfect for the lighter triplet. This difference is to be expected; the higher energy states have the quarks better smeared out so their position information is less important. Many other hadrons can be fit with analogous formulas. See [19] for fits to over 100 other hadrons.

5.3 Mixing

When a charged lepton emits or absorbs a W particle and becomes a neutrino, it can change its generation. The amplitudes for the process defines a unitary

3×3 matrix, the MNS matrix. As with mass, the presence of w^g in the long term propagators has implications for the form of the MNS matrix. But now there are two sets of long term propagators. Instead of a function of the complex variable u_g as in mass, we will look for a function of two complex variables, $U(u_g, v_h)$ with the unitary matrix given by the nine entries $U(w^g, w^h)$.

Since $u_g^3 = v_h^3 = 1$, a general function of these variables can be written in terms of the nine complex constants a_{jk}:

$$U(u,v) = a_{00} + a_{01}v + a_{02}v^2 + a_{10}u + a_{11}uv + a_{12}uv^2 + a_{20}u^2 + a_{21}u^2v + a_{22}u^2v^2. \tag{66}$$

The a_{jk} are the discrete Fourier transform of the unitary matrix $U(w^g, w^h)$.

The long term propagators sum to unity Eq. (46), so we expect that summing over g or h will give a simple result. The simplest possible result is that summing over g gives no dependency on h and vice versa. Adding this requirement to Eq. (66) eliminates the a_{01}, a_{02}, a_{10}, and a_{20} terms leaving five terms:

$$U(u_g, v_h) = a_{00} + a_{11}u_g v_h + a_{22}u_g^2 v_h^2 + a_{12}u_g v_h^2 + a_{21}u_g^2 v_h, \tag{67}$$

this gives:

$$U(g,h) = a_{00} + a_{11}w^{g+h} + a_{22}w^{-(g+h)} + a_{12}w^{g-h} + a_{21}w^{h-g}. \tag{68}$$

The a_{12} and a_{21} terms depend on $g-h$ while the a_{11} and a_{22} terms depend on $g+h$. The a_{00} gives a constant added to all the entries.

Experimental measurements of the mixing matrices are restricted to the magnitudes of the entries. The values for the MNS follow approximately the tribimaximal [20] form. Its probabilities (the squares of the unitary magnitudes) are:

$$P_{MNS} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \tag{69}$$

Since there is a zero entry, the above can be written uniquely as the sum of a non-negative 1-circulant and a non negative 2-circulant: [21]

$$P_{MNS} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{6} \end{pmatrix}. \tag{70}$$

The above form is suggestive. The 1-circulant matrix has entries that depend on $(g-h)$ mod (3), while the 2-circulant entries depend on $(g+h)$ mod (3). We can write the tribimaximal probabilities as functions of $(g-h)$ and $(g+h)$ as follows:

$$P_{MNS,gh} = 1/3 + [\sin(2\pi(g-h)/3) + \sin(2\pi(g+h)/3)]/\sqrt(27). \tag{71}$$

This is a symmetric description of the tribimaximal probabilities.

Given the above split of the tribimaximal probabilities into 1-circulant and 2-circulant portions, it is natural to look for a unitary matrix that splits the same way. A symmetric choice is:

$$
\begin{array}{rcl}
U_{MNS} & = & \frac{\sqrt{+i}}{\sqrt{3}}\begin{pmatrix} 1 & +\sqrt{1/2} & 0 \\ 0 & 1 & +\sqrt{1/2} \\ +\sqrt{1/2} & 0 & 1 \end{pmatrix} \\
& & \frac{\sqrt{-i}}{\sqrt{3}}\begin{pmatrix} 1 & -\sqrt{1/2} & 0 \\ 0 & 1 & -\sqrt{1/2} \\ -\sqrt{1/2} & 0 & 1 \end{pmatrix}
\end{array}
\tag{72}
$$

See [21] for a derivation of the above matrix using the discrete Fourier transform. In the above, the complex phases $\sqrt{\pm i}$, and the magnitudes $\sqrt{1/2}$ are the same as seen in the off diagonal entries for products like $(XY)(YZ)$ seen in Eq. (17). Consequently we expect that this form of the tribimaximal unitary mixing matrix can be derived as a path integral.

The quark mixing matrix, CKM, is simpler than the MNS matrix in that, except for the Cabibbo angle, it is nearly diagonal. The Cabibbo angle is around 2/9 and gives the mixing between the first and second generations. This suggests that a correct analysis of the mixing matrices, in terms of tripled Pauli spin propagators, should use the form $\exp(i(\pi/12 + 2/9 + 2g\pi/3))$ form.

6 Discussion

In ontological theories of quantum mechanics, one attempts to choose which mathematical treatment of quantum mechanics is a model of reality in that the elements of the model correspond directly to elements of Nature. If we choose spin and position as ontological observables, then it is natural for us to desire that they be treated mathematically in a similar way. Svetlichny showed [1] that position can be treated as a sum over products of mutually unbiased bases. The present paper shows that spin can also be treated this way.

The calculation is robust in that the long time propagators G_g^∞ of Eq. (41) do not depend on the details of G_g. For example, one could consider a theory where the paths require all adjacent forms to be different. This would allow paths like $XYZX$ but disallow $XXYZ$. This change is accomplished by putting zeroes on the diagonal of G_g. The result would be that G_g^N and κ would be different, but the discrete Fourier transform of G_g would still have one element larger in magnitude than the others and this element would dominate G_g^N leaving G_g^∞ unchanged.

Mutually unbiased bases are a fundamental object of study in quantum information theory. They encapsulate the essence of the relationship between complementary observables such as position and momentum. To find MUBs at the foundation of position path integrals, spin-1/2, and the particle generations may be more surprising to theorists in elementary particle than quantum information.

The tripled Pauli spin calculations in the present paper are done using the path integral formalism. As such, these calculations are a part of quantum mechanics. However, tripled Pauli spin is not one of the irreducible representations of $SU(2)$ and so there will be some resistance to accepting it as a theory of spin.

In modern elementary particle theory spin arises as a result of examining the irreducible representations of the homogeneous Lorentz group [22], that is, spin-1/2 is one of the few possibilities allowed in the intersection of the special theory of relativity with quantum mechanics. Physics has found this intersection a bountiful place to look for elementary particle models. Despite these successes, there has been some difficulty combining gravitation with quantum mechanics.

Our best theories, when extrapolated to very small distances, predict that space is anything but flat. Lorentz invariance cannot possibly apply at short distances so it should not be used to restrict theories in that regime. Instead, the assumptions of this paper must be judged on the basis of how arbitrary they are, and whether the resulting calculations are compatible with observations.

Situations involving extremely non flat spacetime are available, at least theoretically, in that we can look at the dynamics of black holes. Non rotating black holes exponentially approach spherical symmetry. Presumably, their exponential approach is accomplished by the radiation of elementary particles. Accordingly, L. Motl [23] examined the vibration modes of black holes for various spin cases. In addition to the expected results, he found a spectrum of vibrations he called "tripled Pauli statistics". These were the results of the spin-0 and spin-2 vibration modes. Perhaps they have something to do with tripled Pauli spin.

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