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# Density Operator Theory and Elementary Particles

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**Abstract** Density operator or density matrix formalism is an alternative formulation of quantum mechanics that has several advantages over the usual Hilbert space formulation. Instead of vectors, operators are used to represent quantum states. Applications of the formulation to the elementary fermions are presented.

**Keywords** time · density operator · density matrix · Bohmian mechanics · primitive idempotents · elementary fermions

Almost any introductory quantum mechanics text will show how to convert a normalized wave function  $\psi_a(\mathbf{x}, t)$  or ket  $|a\rangle$  into pure density operator form by the transformations:

$$\begin{aligned}\psi_a(\mathbf{x}, t) &\rightarrow \rho_a(\mathbf{x}, t; \mathbf{x}', t') = \psi_a(\mathbf{x}, t) \psi_a^*(\mathbf{x}', t'), \\ |a\rangle &\rightarrow \rho_a = |a\rangle\langle a|.\end{aligned}\tag{1}$$

The text will go on[1] to note that pure density operators can be generalized to statistical mixtures that cannot be modeled by wave functions or kets. An implication of this definition is that the wave function or ket is the fundamental object which represents the quantum state. From these, the density operator is derived, and then, the only use of the pure density operator is in its providing a core on which to build the statistical generalization.

This paper will assume a reversed ontology; we will treat the pure density operator as fundamental, with the wave function or ket as a derived formulation convenient for mathematical calculations. This first section will define the pure density operators as a subset of the general class of operators, those that are Hermitian projection operators with unit trace (i.e. are Hermitian primitive idempotents).

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The usual generalization of the pure density operators is to loosen the restriction that they be projection operators. The results are interpreted as representing statistical mixtures of quantum states. In the second section we loosen the requirement that pure density operators have unit trace and show that the weak quantum numbers of the quarks and leptons can be related to the permutation group on three elements.

In assuming that the density operator form  $\rho_a(\mathbf{x}, t; \mathbf{x}', t')$  is fundamental we are also invited to reinterpret the meaning of spacetime, particularly its number of dimensions. A natural interpretation is that each spacetime event needs to be described by a position  $\mathbf{x}$ , and two times  $t$  and  $t'$ . The two times gives the absolute time of the event, and the time of the event relative to the observer. In the third section we show that this interpretation gives a natural way of providing a continuous mechanism for wave function collapse.

Adding an extra time coordinate to spacetime events suggests that the geometry of spacetime should be modeled by the Clifford algebra  $C(3, 2)$  rather than the  $C(3, 1)$  or Dirac's gamma matrices. In the fourth section we explore the pure density operators of  $C(3, 2)$ . We propose that these represent preons and compare their quantum numbers with the weak quantum numbers of the quarks and leptons.

## 1 Pure Density Operators

Mathematically, the advantage of pure density operators over wave functions and kets is that all the objects of the theory are of the same type; one deals with operators only. The  $N \times N$  matrices are naturally operators in themselves; they have addition and multiplication defined as usual, and include a zero and one which we will write as 0 and 1. We will write the elements of such a matrix as  $G_{jk}$ . In converting the kets to density operator form, we eliminate the column vectors from our formulation. The formulation's mathematical objects will all consist of  $N \times N$  matrices.

For the wave functions, we will write operators in Green's function form. An operator  $G(\mathbf{x}, t; \mathbf{x}', t')$  acting on a wave function  $\psi(\mathbf{x}, t)$  gives a new wave function  $(G\psi)(\mathbf{x}, t)$  as follows:

$$(G\psi)(\mathbf{x}, t) = \int G(\mathbf{x}, t; \mathbf{x}', t') \psi(\mathbf{x}', t') d^3x' dt', \quad (2)$$

where all our integrals are to be taken over all spacetime. This is a quite general formulation of quantum mechanics. For example, if we were interested in the excited states of hydrogen, the operator we would be concerned with would be the Green's function for Schrödinger's equation for the hydrogen atom. Similarly, the definition of the product of two operators  $G$  and  $H$  is given by:

$$(GH)(\mathbf{x}, t; \mathbf{x}'', t'') = \int G(\mathbf{x}, t; \mathbf{x}', t') H(\mathbf{x}', t'; \mathbf{x}'', t'') d^3x' dt'. \quad (3)$$

Addition is defined by summing the two operators as functions so zero is the zero operator. The unit operator, 1 is defined by the product of four delta

functions, that is,  $1 = \delta(x_1 - x'_1)\delta(x_2 - x'_2)\delta(x_3 - x'_3)\delta(t - t')$ . Thus for wave functions or kets, the pure density operators are elements of an algebra.

A useful complex valued function on operators is the trace. For the  $N \times N$  matrices, this is just the usual sum of the diagonal terms. For an operator  $G(\mathbf{x}, t; \mathbf{x}', t')$  the trace is defined as:

$$\text{tr}(G) = \int G(\mathbf{x}, t; \mathbf{x}, t) d^3x dt. \quad (4)$$

In either case, the trace of a pure density operator is 1. If an operator has been constructed as a pure density operator from a wave function or ket, its trace will automatically be 1. This follows from the normalization of the wave function or ket. Note that the trace is linear,  $\text{tr}(G + H) = \text{tr}(G) + \text{tr}(H)$ , that  $\text{tr}(0) = 0$ , and that  $\text{tr}(1) = N$  where  $N$  is the size of the  $N \times N$  matrix (or infinity for wave functions).

An “idempotent” is an operator  $G$  that satisfies the equation:

$$G^2 = G. \quad (5)$$

In physics these are better known as projection operators. Pure density operators are idempotent but this property is not enough to distinguish them from more general operators. In particular, 0 and 1 are idempotent but are not pure density operators. Since the eigenvalues of the square of an operator are the squares of the eigenvalues of the operator, the eigenvalues of an idempotent operator must be idempotents of the complex numbers, that is, they must be complex numbers satisfying the equation  $\lambda^2 = \lambda$ . Hence the only eigenvalues an idempotent operator may have are 0 and 1.

An idempotent is “primitive” if its trace is one. An equivalent definition, preferred by mathematicians, is that the primitive idempotents are the ones that cannot be written as the sum of two nonzero idempotents. Thus we have that the pure density operators are Hermitian primitive idempotents. In fact, this characterization is both necessary and sufficient. Instead of defining the pure density operators as the operators which can be obtained from a wave function or ket, we can define them as the Hermitian primitive idempotents.

To show that the Hermitian primitive idempotents are the same as the pure density operators we can describe how to obtain the wave function or ket from the pure density operator. This is the transformation opposite to Eq. (1), we will be defining the wave function or ket in terms of the density operator. In doing this, one must note that the wave function and ket formulations require the choice of an arbitrary complex phase. This phase does not appear in pure density operator form consequently we will have a choice to make in the reverse transformation.

Since the trace of a pure density operator is not zero, there must be an element on the diagonal which is nonzero. That is, we must have  $G(\mathbf{x}_k, t_k; \mathbf{x}_k, t_k) \neq 0$  or for the matrices,  $G_{kk} \neq 0$ . Then the wave function can be taken to be a multiple of  $G(\mathbf{x}, t; \text{vec} \mathbf{x}_k, t_k)$  or the column vector through  $G_{kk}$ . Our choice of which  $k$ , after normalization, amounts to a choice of an arbitrary complex

phase:

$$\begin{aligned}\rho &\rightarrow \psi(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t; \mathbf{x}_k, t_k)}{\sqrt{\rho(\mathbf{x}_k, t_k; \mathbf{x}_k, t_k)}}, \\ G &\rightarrow \frac{\text{any non zero column of } G}{\sqrt{\text{Squared magnitude of that column}}}.\end{aligned}\tag{6}$$

Thus one can obtain the wave function and matrix formulations of quantum mechanics from the corresponding pure density operator formulations.

An important concept of wave functions and kets is orthogonality. Two wave functions are orthogonal when their inner product is zero. When this is translated into pure density matrices, we have that two pure density operators,  $G$  and  $H$ , are orthogonal if  $GH = 0$ , and by Hermiticity, we also have  $HG = 0$ . When two operators have this feature we call them annihilators of each other.

A set of mutually orthogonal wave functions or kets is complete if no other orthogonal wave functions or kets exist. Such a set forms a basis set for the Hilbert space. The corresponding concept in operators is a “complete set of mutually annihilating Hermitian primitive idempotents”, or, for short, a “complete set of states.” Such a set sums to unity:

$$\sum_{n=1}^{\infty} G_n = 1.\tag{7}$$

In the algebra of Pauli spin operators, that is, the  $2 \times 2$  complex matrices, the usual example of a complete set of states is spin up and spin down, i.e.  $G = \{(1 + \sigma_z)/2, (1 - \sigma_z)/2\}$ .

If the dimension of the Hilbert space (over the complex numbers) is  $N$ , then the operator space has dimension  $N^2$ . Consequently, a complete set of states is not a basis set for the operator algebra (considered as a vector space of dimension  $N^2$  over the complex numbers). To make it into a basis set for the operator algebra, it helps to first define the concepts of “mutually unbiased bases” for wave functions or kets, and the corresponding concept, “mutually unbiased complete sets of states” for the operator algebra.

Suppose we have two basis sets for the Hilbert space,  $\{|g_j\rangle\}$  and  $\{|h_k\rangle\}$ . If we have that the inner products between  $|g_j\rangle$  and  $|h_k\rangle$  does not depend on  $j$  or  $k$ , then we say that the two basis sets are “mutually unbiased”. The corresponding concept in pure density matrices uses two complete sets of states  $\{G_j\}$  and  $\{H_k\}$ . They are called “mutually unbiased” if  $\text{tr}(G_j H_k)$  does not depend on  $j$  or  $k$ . Mutually unbiased states are particularly important in quantum mechanics in that the Fourier transform, or for kets, the discrete Fourier transform, converts a basis set for the Hilbert space into another basis set that is mutually unbiased with respect to the original. Note that with operators, Fourier transforms are two-sided.

Now we can define a basis set for the operator algebra. Let  $\{G_n\}$  be a complete set  $N$  states, and let  $H$  be one state taken from another complete set  $N$  states that happens to be mutually unbiased with respect to  $\{G_n\}$ . If we multiply  $H$  by the number of states in the basis,  $N$ , we have an operator  $D = NH$ , that is called the “democratic operator” or democratic matrix. The term comes from the fact that when we write  $D$  in the matrix representation that it and  $\{G_n\}$  generate,  $D$  will have all its elements equal to 1. Given such

a  $D$  the following set:

$$\{G_{jk}\} = \{G_j D G_k \mid 1 \leq j, k \leq N\} \quad (8)$$

is a basis set for the operator algebra. One use for mutually unbiased bases is to create a faithful matrix representation of an operator algebra. For example, the Pauli spin matrices are generated by taking  $G = \{(1 + \sigma_z)/2, (1 - \sigma_z)/2\}$  and  $H = (1 + \sigma_x)/2$  so  $D = 1 + \sigma_x$ . Dirac's gamma matrices can be generated similarly.

The diagonal operators of the operator algebra basis set  $\{G_{jj}\}$  are just the usual complete set of states. There are  $N$  of them and they are idempotent. On the other hand, the  $N^2 - N$  off diagonal elements are nilpotent, that is, they square to zero. When these off diagonal elements are applied to wave functions or kets, they act as raising or lowering operators. That is, the operator  $G_{jk}$  converts the state  $\psi_k$  to  $\psi_j$  and zeroes any other state. While the diagonal states are Hermitian, these raising and lowering operators are non Hermitian. Raising and lowering operators carry arbitrary complex phases, but the above raising and lowering operators are defined entirely from pure density operators which have no arbitrary complex phase. Instead, raising and lowering operators get their arbitrary complex phases from the arbitrary choice of  $D = NH$ .

No matter what wave function or ket we choose, the conversion to an operator will be an Hermitian primitive idempotent. We can loosen these restrictions to create interesting generalizations of pure density operators. By far the most common generalization is used to obtain the mixed density operators. For these, we take a partition of unity, that is, a collection of non negative real numbers  $\{\alpha_j\}$  that sum to unity (and represent probabilities), and multiply a complete set of states  $G_j$  by these real numbers. The result is a mixed density operator:

$$\Sigma_j \alpha_j G_j. \quad (9)$$

Since the trace is linear, mixed density operators still have a trace of 1, that is, they are primitive. And they are still Hermitian. But, in general, they are not idempotent. We will not be concerned with this generalization of pure density operators.

If we give up the requirement that the operators representing states be primitive, but keep Hermitian and idempotent, we end up a representation that seems to allow multiple states to be represented in the same operator, so long as they are orthogonal. Given a basis set of  $N > 2$  states  $\{G_j\}$ , we can add two, say  $G_1 + G_2$ . This sum will still be idempotent, but since the trace is linear the trace will be 2. This interpretation is defective in that it does not use the correct phase space for multiple particle states; there is no way to adjust the symmetrization.

A two particle wave function differs from a single particle wave function in that the position variable is duplicated, one for each of the two particles. For example,  $\psi(\mathbf{u}, \mathbf{v}, t)$ , where  $\mathbf{u}$  and  $\mathbf{v}$  are two 3-d vectors. This is the object on which we are to apply symmetrization or anti-symmetrization. Now suppose that the two particles are so closely bound that, at the level of approximation we would like to model, their positions are identical. That is, we suppose that  $\rho$  is non zero only for  $\mathbf{u} = \mathbf{v}$ . Then, in this approximation, we may

as well abbreviate the wave function as  $\psi(\mathbf{u}, t)$ . This suggests that we can consider generalizations of the idempotency requirement when looking for preon theories. We will explore this idea further in the second section of this paper.

A third generalization is to give up Hermiticity, but to still require the states to be primitive idempotents. Using  $G_1$ ,  $G_2$  and  $D$  as before, an example of a non Hermitian primitive idempotent is  $G_1 + G_1 D G_2$ . Its Hermitian conjugate is  $G_1 + G_2 D G_1$  which is different as they correspond to different combinations of basis elements of the operator algebra, and therefore are different. In the Pauli algebra, an example of a non Hermitian primitive idempotent is:

$$(1 + \sigma_z)/2 + (1 + \sigma_z)/2(1 + \sigma_x)(1 - \sigma_z)/2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (10)$$

In the above,  $D = NH = 1 + \sigma_x$ , while  $(1 \pm \sigma_z)/2$  are the spin up and spin down states.

A non Hermitian primitive idempotent  $G$  can always be written uniquely as a complex multiple of the product of two Hermitian primitive idempotents. The non Hermiticity means that the operator does not have the same left side eigenvectors as right side. Since the operator is primitive and idempotent, there is exactly one left eigenvector,  $\psi_L$  with eigenvalue 1, and another right eigenvector  $\psi_R$ , with eigenvalue 1. These two eigenvectors carry an arbitrary complex phase, but it is eliminated when they are converted to pure density operators  $L$  and  $R$  so these are uniquely defined. The product  $LR$  has the same eigenvectors and eigenvalues, both left and right, as  $G$ , and so is a complex multiple of  $G$ , or is zero. But the product cannot be zero as  $G^2 = G \neq 0$ .

These non Hermitian primitive idempotents arise naturally when one ignores time dependence in pure density operators. For example, suppose that  $\rho(\mathbf{x}, t; \mathbf{x}', t')$  defines the wave function for an electron state that is initially polarized with spin in the  $(0, 0, 1)$  direction but finally polarized with spin in the  $(1, 0, 0)$  direction. This could happen because the wave packet passes through a region where an interaction influences the spin to rotate.

If we were concerned with the initial and final states of the electron, but were not concerned with the details of what happened in between, we could naively represent this situation with a non Hermitian primitive idempotent. In this case the right, or initial, side would be  $R = (1 + \sigma_z)/2$  and the left, or final, side would be  $L = (1 + \sigma_x)/2$ . Their product has a trace of  $1/2$ , consequently the non Hermitian primitive idempotent would be  $(1 + \sigma_x)(1 + \sigma_z)/2$ .

However, be warned that this definition is not internally consistent! The reverse product,  $(1 + \sigma_z)(1 + \sigma_x)/2$  left multiplying  $(1 + \sigma_x)(1 + \sigma_z)/2$  gives twice the expected Hermitian primitive idempotent  $(1 + \sigma_z)/2$ . The problem with this use of non Hermitian primitive idempotents is in the blind insistence on their normalization to idempotency. Since their inputs and outputs are incompatible states, one should instead require normalization by requiring primitive idempotency when they are multiplied by their Hermitian conjugate.

A related generalization of Hermitian primitive idempotents was explored by Julian Schwinger in the 1950s.[4,5] He called it the “measurement algebra,” and it is the subject of one of the most elegant undergraduate introductions to quantum mechanics, [6]. Nevertheless, it is sufficiently obscure that [1] overlooked it in their list of the formulations of quantum mechanics. Schwinger does not use the language of generalized Hermitian primitive idempotents or density matrices, but his formulation is built from idempotents and so is related to pure density operators and can be explained in that context.

In the measurement algebra, an operator (or algebra element) represents the content of a beam of elementary particles. Both intensity and which polarizations are present. For example, if a beam of spin-1/2 particles has unit intensity with spin up, then it is represented by the primitive idempotent,  $(1 + \sigma_z)/2$ . A beam with twice this intensity would be represented by  $2(1 + \sigma_z)/2$ . If this beam picks up a complex phase  $\eta$  and has a strength of three units, with the same particle content and no change to the polarization, then the representation becomes  $3 \exp(i\eta)(1 + \sigma_z)/2$ . In this formulation, the operator 0 represents a beam with no particles, while the 1 operator represents a full strength beam with all particle content at unit level.

The “elementary measurements” of the measurement algebra are elements of the algebra that satisfy a set of definitions that are equivalent to the definitions of Hermitian primitive idempotents. These represent unit strength beams where the particle content has been completely determined, spin, charge, etc., for example, positrons with spin oriented in the  $+y$  direction. One can add these together; this corresponds to uniting the beam with another beam with the appropriate properties. Since beams can be multiplied by complex constants, interference can occur.

If one begins with a fully unpolarized beam of unit strength, i.e. 1, one obtains an elementary measurement (Hermitian primitive idempotent) by sending the beam through a “complete” beam splitter, that is, a beam splitter that separates particles according to their mass, charge, spin, and any other properties needed to define a complete set of quantum observables. The resulting beam will have all its particles identical and corresponds to an elementary measurement. In this way, Schwinger combines the concept of experimental equipment and quantum states; an operator represents both the beam of particles that comes from it and also the separation equipment itself.

A Stern-Gerlach apparatus consists of a severely inhomogeneous magnetic field. If the apparatus is to separate the beam in the  $z$  direction, then the magnetic field inhomogeneity must be in the  $z$  direction. In this sense, the Hermitian primitive idempotents represent three physical objects, the quantum state of the particles in the beam, the beam splitter, and also the geometry of the beam splitter. Putting these together, Schwinger’s invention provides a natural way to write the quantum states in terms of their geometry. This will be the topic of the third section.

## 2 Weak Quantum Numbers

In this section we expand the definition of pure density operators by allowing them to violate the usual requirement that they be primitive (i.e. have unit trace). We will be doing this in the context of a model of preons that are so deeply bound that we can assume that the positions of the preons are always identical. Thus we avoid the need to decide whether they are fermions or bosons and correctly symmetrize their wave function.

As an example of this method, consider the total spin of a helium nucleus (two protons and two neutrons) along with zero, one, or two electrons, in their lowest energy states. The spin of the nucleus is zero, and since the particles are to be in their lowest energy states there is no orbital angular momentum. Four states appear, the neutral helium atom, a singly ionized helium atom with spin up or down, and a doubly ionized helium atom (nucleus). These four states form, respectively, a spin singlet, a spin doublet, and another spin singlet.

The pure density operator for a single electron is of the form  $(1 + \sigma_{\mathbf{u}})/2$  where  $\mathbf{u}$  is a 3-dimensional vector specifying the orientation of the spin. We will take the traditional  $z$  axis,  $\mathbf{u} = (0, 0, 1)$ , so the complete set of states is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (11)$$

The above are a representation of the spin doublet. Each of them is primitive.

If we loosen the restriction on primitivity, we get two more states in the Pauli algebra, 0 and 1:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (12)$$

We would like to use 0 to represent the situation with no electron, and 1 to represent the situation with two electrons with opposite spin. Adding these to the usual spin-1/2 states, this gives us a total of four possible electron states, we will call these four states “non primitive idempotents”. The trace gives the number of electrons present, 0, 1, or 2.

The operator for spin in the  $z$  direction is given by  $\sigma_z$ :

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (13)$$

To compute the expected value of spin in the  $z$  direction we take the trace of the product of  $S_z$  with our four states. Note that all four of the non primitive idempotents are eigenstates (i.e. double sided eigenvectors) of this operator, and that their (double sided) eigenvalues are 0,  $+\hbar/2$ ,  $-\hbar/2$ , and 0, as is correct.

Suppose that  $Q$  is any operator that corresponds to a measurement that can be made of the helium atom and which measurement gives values proportionate to the number of spin-up and spin-down electrons present, i.e.  $Q$  has expectation values  $q$  of the form

$$q = N_u q_u + N_d q_d, \quad (14)$$



where  $N_u$  and  $N_d$  are the number of up and down electrons (i.e. 0 or 1), and  $q_d$  and  $q_u$  are the values. Then the trace of  $Q$  will give the correct expectation values for all four of the states. As an example, the total electron charge (which operator will be  $e1$  where  $e$  is the electron charge) follows this rule and will be correctly modeled for the zero and two electron states. Preon models of the quarks and leptons will naturally wish to obtain the correct electric charges and weak quantum numbers of these particles. Each preon type, if present, should modify these quantum numbers by a fixed amount and so this is a natural method to use to look for preon models of the quarks and leptons.

In the above example, we found a complete set of Hermitian idempotents (primitive and non primitive) for the  $2 \times 2$  complex matrices. The primitive idempotents can be defined solely from algebraic relations of the operators; our result is apparently unrelated to the fact that the Hermitian primitive idempotents happened to form a complete basis for a representation of  $SU(2)$ .

To see this in general, let's solve for the idempotency condition with a general, Hermitian,  $2 \times 2$  complex matrix  $A$ :

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}. \quad (15)$$

The condition that  $A^2 = A$  resolves into three coupled quadratic equations:

$$\begin{aligned} a &= a^2 + b^*b, \\ b &= ab + bc, \\ c &= c^2 + b^*b. \end{aligned} \quad (16)$$

There are two discrete solutions to the above corresponding to  $A = 0$  and  $A = 1$ , along with a manifold of solutions that can be parameterized by the unit vectors  $\mathbf{u}$ . They correspond to the usual pure density operators with spin in the  $\mathbf{u}$  direction,  $(1 + \sigma_u)/2$ .

However, we are interested only in the spin up and down cases. For these  $b = 0$  and our quadratic equations could have been reduced to just two:

$$\begin{aligned} a &= a^2, \\ c &= c^2. \end{aligned} \quad (17)$$

The above is trivial, but it does define an algebra. It is the algebra of two element vectors  $(a, c)$ , where addition is as usual and multiplication is term wise,  $(a, c)(a', c') = (aa', cc')$ .

Each finite group with  $M$  elements defines a multiplication on vectors with  $M$  elements. To show how this is done, it's useful to reanalyze the helium atom example more geometrically. Examining the definitions of the Pauli spin matrices, let us rewrite the real values  $a, c$ , and the complex value  $b$  of Eq. (16) in terms of four real parameters  $u, x, y$ , and  $z$  defined as follows:

$$\begin{aligned} a &= u + z, \\ b &= x + iy, \\ c &= u - z. \end{aligned} \quad (18)$$

After the substitution, the coupled quadratic equations of Eq. (16) become more elegant:

$$\begin{aligned} u &= u^2 + x^2 + y^2 + z^2, \\ x &= 2ux, \\ y &= 2uy, \\ z &= 2uz. \end{aligned} \tag{19}$$

The vector  $(x, y, z)$  defines the Bloch sphere. In that we are only considering spin in the  $z$  direction, we will put  $x = y = 0$ . This simplifies the equations even further, and we can rewrite them in a more suggestive form:

$$\begin{aligned} u &= uu + zz, \\ z &= uz + zu. \end{aligned} \tag{20}$$

It is easy to verify that the solutions to the above equations are the two singlets  $(u, z) = (0, 0)$ , and  $(1, 0)$ , and the doublet  $(1/2, \pm 1/2)$ . Thus  $u$  is proportional to the total charge on the electrons while  $z$  is proportional to the total electron spin.

In the simplified equations Eq. (20), all the possible products between  $u$  and  $v$  are used on the right hand side, two in the equation for  $u$  and two in the equation for  $z$ . Now think of  $u$  and  $z$  as permutations of two objects;  $u$  being the identity and  $z$  being the swap. The above quadratic equations can be interpreted as a prescription for how to multiply the group elements. That is, the group multiplication table is:

$$\begin{array}{c|cc} * & u & z \\ \hline u & u & z \\ z & z & u \end{array} \tag{21}$$

There are two ways of obtaining  $u$ , as a product of two group elements, that is,  $u = uu$  and  $u = zz$ . This is the first line of Eq. (20). Similarly there are two ways of obtain  $z$  and these give the second line.

There are two lessons here. The first is that the above process can be repeated with any finite group. The result will be a set of coupled quadratic equations. We can then solve the equations and consider each solution as a set of quantum numbers for a not necessarily primitive idempotent. The second lesson is that the finite group we obtain gives a little information about the Clifford algebra. In the above example, the finite group is the group with two elements; this corresponds to the fact that  $\sigma_z^2 = 1$ .

The first generation left and right handed elementary fermions, and their antiparticles, have the following weak hypercharge ( $t_0$ ) and weak isospin ( $t_3$ ) quantum numbers, particles on the left and anti-particles on the right:

$$\begin{array}{c|cc|cc} & t_0 & t_3 & & t_0 & t_3 \\ \hline \nu_L & -1 & +1/2 & \bar{\nu}_R & +1 & -1/2 \\ \nu_R & 0 & 0 & \bar{\nu}_L & 0 & 0 \\ d_L & +1/3 & -1/2 & \bar{d}_R & -1/3 & +1/2 \\ d_R & -2/3 & 0 & \bar{d}_L & +2/3 & 0 \\ e_L & -1 & -1/2 & \bar{e}_R & +1 & +1/2 \\ e_R & -2 & 0 & \bar{e}_L & +2 & 0 \\ u_L & +1/3 & +1/2 & \bar{u}_R & -1/3 & -1/2 \\ u_R & +4/3 & 0 & \bar{u}_L & -4/3 & 0 \end{array} \tag{22}$$

The quantum numbers of the other generations are the same. Note that each pair of quantum numbers appears twice, once for a particle and once again, negated, for an anti-particle. Following the helium atom example, we seek a finite group that has the above quantum numbers as a solution to its associated set of coupled quadratic equations.

As the size of the finite group increases, the difficulty in solving its coupled quadratic equations increases very steeply. Fortunately, the solution turns out to be the permutation group on three elements. In defining this group, we will permute the three colors red, green, and blue abbreviated  $R$ ,  $G$ , and  $B$ . We define the six group elements to be  $I$ ,  $J$ ,  $K$ ,  $R$ ,  $G$ , and  $B$  with their actions on  $RGB$  as follows:

$$\begin{array}{c|ccc}
 * & R & G & B \\
 \hline
 I & R & G & B \\
 J & G & B & R \\
 K & B & R & G \\
 R & R & B & G \\
 G & B & G & R \\
 B & G & R & B
 \end{array} \quad (23)$$

That is,  $I$  is the identity,  $J$  and  $K$  are the cyclic permutations, while  $R$ ,  $G$ , and  $B$  are the swaps. The permutatoin group multiplication table is then:

$$\begin{array}{c|cccccc}
 & I & J & K & R & G & B \\
 \hline
 I & I & J & K & R & G & B \\
 J & J & K & I & B & R & G \\
 K & K & I & J & G & B & R \\
 R & R & G & B & I & J & K \\
 G & G & B & R & K & I & J \\
 B & B & R & G & J & K & I
 \end{array} \quad (24)$$

And the associated six coupled quadratic equations are:

$$\begin{aligned}
 I &= I^2 + JK + KJ + R^2 + G^2 + B^2, \\
 J &= IJ + JI + K^2 + RG + GB + BR, \\
 K &= IK + J^2 + KI + RB + GR + BG, \\
 R &= IR + JG + KB + RI + GK + BJ, \\
 G &= IG + JB + KR + RJ + GI + BK, \\
 B &= IB + JR + KG + RK + GJ + BI.
 \end{aligned} \quad (25)$$

The assignment of the values  $I$ ,  $J$ ,  $K$ ,  $R$ ,  $G$ , and  $B$  to the weak quantum numbers cannot be arbitrary. Since  $I$  is the finite group's identity, it will have to give the  $U(1)$  quantum number  $t_0$ . The generators of  $SU(2)$  square to unity, and this corresponds to the group elements  $R$ ,  $G$ , and  $B$ , so these will be related to the quantum number  $t_3$ . The solutions will have  $t_0 = 2I$  and  $t_3 = R + G + B$ .

As a first step in solving these equations, we rewrite them as an equivalent set of six equations:

$$\begin{aligned}
I &= I^2 + 2JK + (R^2 + G^2 + B^2), \\
(R + G + B)^2 &= (I + J + K)(1 - (I - J - K)), \\
0 &= (J - K)(1 + J + K - 2I), \\
(1 - 3I + (I + J + K))R &= (R + G + B)(J + K), \\
(1 - 3I + (I + J + K))G &= (R + G + B)(J + K), \\
(1 - 3I + (I + J + K))B &= (R + G + B)(J + K).
\end{aligned} \tag{26}$$

Choosing  $I = 1/2$  and  $J = -K$  solves the last four of these equations. The two remaining equations reduce to:

$$\begin{aligned}
1/2 &= \pm(R + G + B), \\
J^2 &= -1/8 + (R^2 + G^2 + B^2)/2.
\end{aligned} \tag{27}$$

Since we have solved four equations with only two assignments, the solution space will be at least a 2-manifold. We will parameterize the solutions with complex numbers  $\alpha$ , and  $\beta$ . Eventually we find that we can write four 2-manifolds of solutions:

$I$	$J$	$K$	$R$	$G$	$B$	
$1/2$	$+\gamma$	$-\gamma$	$+1/6 + \alpha$	$+1/6 + \beta$	$+1/6 - \alpha - \beta$	
$1/2$	$-\gamma$	$+\gamma$	$+1/6 + \alpha$	$+1/6 + \beta$	$+1/6 - \alpha - \beta$	
$1/2$	$+\gamma$	$-\gamma$	$-1/6 + \alpha$	$-1/6 + \beta$	$-1/6 - \alpha - \beta$	
$1/2$	$-\gamma$	$+\gamma$	$-1/6 + \alpha$	$-1/6 + \beta$	$-1/6 - \alpha - \beta$	(28)

where  $\gamma = \sqrt{\alpha^2 + \beta^2 + \alpha\beta - 1/12}$ . The above solutions share  $t_0 = 2I = +1$ , and have  $t_3 = R + G + B = \pm 1/2$ . These are the weak quantum numbers of the  $\bar{\nu}_R$  and  $\bar{e}_R$ .

Eliminating the case “ $I = 1/2$  and  $J = -K$ ”, there are 10 discrete solutions. Six of these show up as two triplets:

$I$	$J$	$K$	$R$	$G$	$B$	
$1/3$	$w^{+n}/3$	$w^{-n}/3$	$0$	$0$	$0$	
$2/3$	$-w^{+n}/3$	$-w^{-n}/3$	$0$	$0$	$0$	(29)

where  $w = \exp(2i\pi/3)$  and  $n = 0, 1, 2$ . All six of these solutions have weak isospin zero. The two triplets differ in weak hypercharge with  $t_0 = 2I = +2/3$  and  $t_0 = 2I = +4/3$ . These are the weak quantum numbers of the  $d_R$  and  $u_R$ .

The remaining four discrete solutions have different combinations of weak hypercharge and weak isospin:

$I$	$J$	$K$	$R$	$G$	$B$	
$0$	$0$	$0$	$0$	$0$	$0$	
$1$	$0$	$0$	$0$	$0$	$0$	
$1/6$	$1/6$	$1/6$	$-1/6$	$-1/6$	$-1/6$	
$1/6$	$1/6$	$1/6$	$+1/6$	$+1/6$	$+1/6$	(30)

The first two of these solutions have weak isospin 0 and weak hypercharge 0 and 2. These are the quantum numbers of the  $\nu_R$  (or  $\bar{\nu}_L$ ) and the  $\bar{e}_L$ . The

last two solutions share weak hypercharge  $1/3$  and have weak isospin  $\pm 1/2$ ; these are the quantum numbers of the  $d_L$  and  $u_L$ .

The complete set of solutions to the 6 coupled quadratic equations, and their assignment to the first generation fermions are as follows:

	$I$	$J$	$K$	$R$	$G$	$B$	
$\bar{\nu}_L/\nu_R$	0	0	0	0	0	0	
$d_L$	1/6	1/6	1/6	-1/6	-1/6	-1/6	
$u_L$	1/6	1/6	1/6	+1/6	+1/6	+1/6	
$\bar{d}_L$	1/3	$w^{+n}/3$	$w^{-n}/3$	0	0	0	(31)
$\bar{\nu}_R$	1/2	$\pm\gamma$	$\mp\gamma$	$-1/6 + \alpha$	$-1/6 + \beta$	$-1/6 - \alpha - \beta$	
$\bar{e}_R$	1/2	$\pm\gamma$	$\mp\gamma$	$+1/6 + \alpha$	$+1/6 + \beta$	$+1/6 - \alpha - \beta$	
$u_R$	2/3	$-w^{+n}/3$	$-w^{-n}/3$	0	0	0	
$\bar{e}_L$	1	0	0	0	0	0	

where  $w$  and  $\gamma$  are as above.

This result is lacking in that we have three different choices each for  $\bar{d}_L$  and  $u_R$ , and an infinite number for  $\bar{\nu}_R$  and  $\bar{e}_R$ . We can think of these extra solutions as being in analogy with the non Hermitian solutions to the helium problem or to the unused elements on the Bloch sphere.

A natural representation of the permutation group uses matrices of zeroes and ones. For the even permutations, we have:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (32)$$

No two of these three matrices is non zero in the same position. Consequently, we can multiply each by its corresponding complex number  $I$ ,  $J$ , and  $K$  and then assemble the resulting three matrices them into a single complex matrix  $P_0$ :

$$P_0 = \begin{pmatrix} I & J & K \\ K & I & J \\ J & K & I \end{pmatrix}. \quad (33)$$

This is a 1-circulant matrix. That is, each row is the same as the previous row rotated one position to the right. Similarly, we can assemble the odd permutations into a matrix  $P_1$ :

$$P_1 = \begin{pmatrix} R & B & G \\ B & G & R \\ G & R & B \end{pmatrix}, \quad (34)$$

a 2-circulant matrix. These matrices are an amalgamation of the matrices we'd use to represent the finite group, but we're using them with complex coefficients so they're a little more than that.

The 1-circulant  $3 \times 3$  matrices form a subalgebra of the  $3 \times 3$  matrices; the product or sum of any two such matrices is a matrix of the same sort. Products of two 2-circulant matrices are a 1-circulant, and the product of a 1-circulant and a 2-circulant is 2-circulant. These are the same rules that

apply to the diagonal (1-circulant) and off-diagonal (2-circulant) elements of a  $2 \times 2$  matrix. Consequently, we can assemble  $P_0$  and  $P_1$  into a  $6 \times 6$  matrix:

$$P = \left( \begin{array}{ccc|ccc} I & J & K & R & B & G \\ K & I & J & B & G & R \\ J & K & I & G & R & B \\ \hline R & B & G & I & J & K \\ B & G & R & K & I & J \\ G & R & B & J & K & I \end{array} \right) \quad (35)$$

The six coupled equations Eq. (25) are defined by  $P^2 = P$ .

Matrices of this form are a subalgebra of the  $6 \times 6$  complex matrices. That is, they include 0 and 1, and are closed under negation, addition and multiplication. They can be thought of as defining a form of multiplication that operates between two 6-element complex vectors.

These  $6 \times 6$  matrices give a natural definition of ‘‘Hermiticity’’ to the solutions to the 6 coupled quadratic equations of Eq. (25). We will say that a solution is Hermitian if the related  $6 \times 6$  matrix is Hermitian. Examining Eq. (35), we find Hermiticity requires that  $I$ ,  $R$ ,  $G$ , and  $B$  must be real and that  $J$  and  $K$  must be complex conjugates of each other.

Applying this definition of Hermiticity to the 8 classes of solutions given in Eq. (31), we find that most of the fermions are now represented by a unique Hermitian solution. The assignments of the elementary particles are now:

	$I$	$J$	$K$	$R$	$G$	$B$	
$\bar{\nu}_L/\nu_R$	0	0	0	0	0	0	
$d_L$	1/6	1/6	1/6	-1/6	-1/6	-1/6	
$u_L$	1/6	1/6	1/6	+1/6	+1/6	+1/6	
$\bar{d}_L$	$w^{+n}/3$	$w^{-n}/3$	1/3	0	0	0	
$\bar{\nu}_R$	1/2	$\pm i\sqrt{3}/6$	$\mp i\sqrt{3}/6$	-1/6	-1/6	-1/6	
$\bar{e}_R$	1/2	$\pm i\sqrt{3}/6$	$\mp i\sqrt{3}/6$	+1/6	+1/6	+1/6	
$u_R$	2/3	$-w^{+n}/3$	$-w^{-n}/3$	0	0	0	
$\bar{e}_L$	1	0	0	0	0	0	

(36)

These assignments are unique up to the choice of the cubed root of unity  $w$ . The simplicity of this derivation suggests that the quarks and leptons are composites with the permutation group on three elements being involved in their structure.

The permutation group on three elements also has applications to the generation structure of the leptons. Let  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\theta$  be four real parameters. Define the six real numbers  $I$ ,  $J$ ,  $K$ ,  $R$ ,  $G$ , and  $B$  by:

$$\begin{aligned} 3I &= \cos(\gamma) + 2 \cos(\theta) \cos(\alpha), \\ 3J &= \cos(\gamma) + 2 \cos(\theta) \cos(\alpha + 2\pi/3), \\ 3K &= \cos(\gamma) + 2 \cos(\theta) \cos(\alpha + 4\pi/3), \\ 3R &= \sin(\gamma) + 2 \sin(\theta) \cos(\beta), \\ 3G &= \sin(\gamma) + 2 \sin(\theta) \cos(\beta + 2\pi/3), \\ 3B &= \sin(\gamma) + 2 \sin(\theta) \cos(\beta + 4\pi/3). \end{aligned} \quad (37)$$

Then the following matrix is unitary:

$$U(\alpha, \beta, \gamma, \theta) = \begin{pmatrix} I & J & K \\ K & I & J \\ J & K & I \end{pmatrix} + i \begin{pmatrix} R & B & G \\ B & G & R \\ G & R & B \end{pmatrix} \quad (38)$$

This form for a unitary matrix is somewhat unusual in that the sum of the entries in any row or column is the same,  $\exp(i\gamma)$ . What's more, an arbitrary  $3 \times 3$  unitary matrix can be brought into the above form by multiplying its rows and columns by appropriate complex phases. In other words, adding the phases for the rows and columns turns the above into an elegant parameterization of all  $3 \times 3$  unitary matrices.

The MNS or PMNS mixing matrix for the leptons is thought to be a unitary  $\times 3$  matrix. The squared magnitudes of its elements have been experimentally determined to be close to a form known as tribimaximal:

$$\begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix}. \quad (39)$$

There are many unitary matrices that are compatible with the above. On putting it into the form Eq. (38) one finds the simple result:

$$\begin{pmatrix} \sqrt{1/3} & \sqrt{1/6} & 0 \\ 0 & \sqrt{1/3} & \sqrt{1/6} \\ \sqrt{1/6} & 0 & \sqrt{1/3} \end{pmatrix} + i \begin{pmatrix} \sqrt{1/3} & \sqrt{1/6} & 0 \\ \sqrt{1/6} & 0 & \sqrt{1/3} \\ 0 & \sqrt{1/3} & \sqrt{1/6} \end{pmatrix} \quad (40)$$

That the permutation group on three elements has use in the generation structure of the fermions is not surprising; it is related to the discrete Fourier transform on three numbers. These subjects are to be more deeply discussed in a later paper by Marni Sheppeard and this author.

### 3 Wave Function Collapse

Pure density operators such as  $\rho(\mathbf{x}, t; \mathbf{x}', t')$  are complex functions defined on pairs of spacetime events. If  $\rho$  is to be the fundamental object, this seems to be telling us something different about the structure of spacetime; events must be considered relative to other events.

A classical wave function depends only on one copy of spacetime. This is the character of the traditional wave functions of quantum mechanics,  $\psi(\mathbf{x}, t)$ . To make a pure density operator appear in this fashion we have to pick some fixed point  $(\mathbf{x}_k, t_k)$  and look at  $\rho$  with respect to that vantage point. In this section we will discuss a possible physical meaning for the vantage point.

To put the problem into human scale, let's use units of years. Suppose that in 2005 an experimenter decides to run an experiment in 2007. He derives a wave function  $\psi(\mathbf{x}, t)$  to represent the particle. To compute the probability density for a position measurement of the particle he makes the substitution  $t = 2007$  into the wave function and obtains a density function  $|\psi(\mathbf{x}, 2007)|^2$ .

To translate this situation into pure density operator form we first create  $\rho(\mathbf{x}, t; \mathbf{x}', t')$ . From the wave function theory we know that the probability density will be obtained by substituting 2007 into both the time coordinates and making the space coordinates equal, i.e.  $\rho(\mathbf{x}, 2007; \mathbf{x}, 2007)$ . But in the pure density operator formalism this seems rather arbitrary. Why should the pure density operator depend on two copies of spacetime but then, when you use it, you end up making them equal?

On the other hand, we can take any spacetime point  $(\mathbf{x}_k, t_k)$  where  $\rho$  is nonzero in its second coordinate and from this obtain the wave function  $\psi$  and from that get the probability density. Assuming that the particle is not destroyed or created we could put any time in the  $t'$  slot.

In the context of a 2005 experimenter looking at a wave function for an experiment to be performed in 2007, the natural choice for  $t'$  is 2005. This gives us  $\rho(\mathbf{x}, 2007; \mathbf{x}', 2005)$ . The probability density for the 2007 experiment is:

$$\begin{aligned} P_{2007}(\mathbf{x}; 2005) &= \int |\rho(\mathbf{x}, 2007; \mathbf{x}', 2005)|^2 d^3x', \\ &= \int \rho(\mathbf{x}, 2007; \mathbf{x}', 2005) \rho^*(\mathbf{x}, 2007; \mathbf{x}', 2005) d^3x', \end{aligned} \quad (41)$$

where the 2005 indicates that this is the probability density as calculated in 2005. The formula is natural in that it is the squared magnitude of the pure density operator, after integrating out the dependency on the extra time coordinate.

Now suppose that it is 2009 and the results of the experiment have been published. The wave function has collapsed. Instead of being described by a probability density  $P(\mathbf{x})$ , it is now described by a fixed position,  $\mathbf{x}_{2007}$ . To write this in a form compatible with a probability density, we can use Dirac's delta function:

$$P_{2007}(\mathbf{x}; 2009) = \delta^3(\mathbf{x} - \mathbf{x}_{2007}). \quad (42)$$

To put this into compatibility with Eq. (41) we need to have

$$\rho(\mathbf{x}, 2007; \mathbf{x}', 2009) = \sqrt{\delta^3(\mathbf{x} - \mathbf{x}_{2007})} \psi_{2009}(\mathbf{x}'), \quad (43)$$

where  $\psi_{2009}(\mathbf{x}')$  is an arbitrary, normalized wave function as it will be integrated over. Of course for this to work  $\rho$  has to be generalized; instead of being generated from a wave function  $\psi$ , it will be more complicated. The generalization is not Hermitian.

In terms of making the calculating the probability density at  $t = 2007$ , the above gives a method that can be arranged to continuously deform a wave function into a classical particle position. The rule is that if  $t \ll t'$ , then  $\rho(\mathbf{x}, t; \mathbf{x}', t')$  is a wave function as a function of  $\mathbf{x}$ , while if  $t \gg t'$  then  $\rho$  describes a particle position (square root of a delta function) as a function of  $\mathbf{x}$ . This use of the pure density operator may provide an ontology for classical stochastic versions of quantum mechanics such as those of Edward Nelson[2] or Michele Pavon[3].



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## 4 Primitive Idempotents of $C(3, 2)$

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