Operator Guide

to the

Standard Model

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Snuarks and Anions,

for the Masses

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For Mom and Dad

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Preface

> The chance is high that the truth lies in the fashionable direction. But, on the off-chance that it is in another direction—a direction obvious from an unfashionable view of field theory–who will find it? Only someone who has sacrificed himself by teaching himself quantum electrodynamics from a peculiar and unfashionable point of view; one that he may have to invent for himself.
>
> Richard Feynman,
> Stockholm, Sweden,
> December 11, 1965

As I write this in October 2006 particle physics is in trouble. Two books stand out on the physics best sellers list: Lee Smolin's *The Trouble With Physics*, and Peter Woit's *Not Even Wrong*. These books show that mankind's centuries long effort to understand the nature of the world has come to a quarter century long pause. The methods that worked between 1925 and 1980 have not succeeded in pushing back the frontier since then. This book defines an alternative path, a path that may once again allow nature's mysteries to unfold to us.

Lee Smolin writes, "When the ancients declared the circle the most perfect shape, they meant that it was the most symmetric: Each point on the orbit is the same as any other. The principles that are the hardest to give up are those that appeal to our need for symmetry and elevate an observed symmetry to a necessity. Modern physics is based on a collection of symmetries, which are believed to enshrine the most basic principles." In this book we will reject symmetries as the most basic principle and instead look to geometry, more specifically, the geometric algebra[1] of David Hestenes. But instead of applying geometric algebra (a type of Clifford algebra) to spinors, we will be applying it to density operator states.

The geometric algebra is elegant and attractive and several authors have applied it to the internal symmetries of the elementary particles. [2] This book has the advantage over previous efforts in that it derives the relationship between the quarks and leptons, the structure of the generations, and provides exact formulas for the lepton masses. On the other hand, this book suffers from the disadvantage of requiring a hidden dimension and that the geometric

algebra be complex. We will attempt to justify these extensions; in short, they are required because the usual spacetime algebra is insufficiently complicated to support the observed standard model particles.

This book is intended as a textbook for graduate students and working physicists who wish to understand the density operator foundation for quantum mechanics. The density operator formalism is presented as an alternative to the usual Hilbert space, or state vector, formalism. In the usual quantum mechanics textbooks, density operators (or density matrices) are derived from spinors. We reverse this, and derive spinors from the density operators. Thus density operators are at least equal to spinors as candidate foundations for quantum mechanics. But we intend on showing more; that the density operator formalism is vastly superior.

In the state vector formalism, one obtains a density operator by multiplying a ket by a bra: $\rho = |A\rangle\langle A|$. Thus a function that is linear in spinors becomes bilinear in density operators. And a function that is linear in density operators becomes non linear when translated into spinor language. This means that some problems that are simple in one of these formalisms will become nonlinear problems, difficult to solve, in the other. To take advantage of both these sorts of problem solving, we must have tools to move back and forth between density operator and state vector form. While most quantum textbooks provide no method of obtaining spinors from density operators, we will, and we will show how to use these methods.

The standard model of the elementary particles has been very good at predicting the results of particle experiments but it has such a large number of arbitrary constants that it has long been expected that it would be eventually replaced with a deeper theory. Dr. Woit's book describes the attraction and ultimate disappointment of string theory. The attraction was the promise of a theory with no need for all those constants; the disappointment was that there were 10^{500} possible quantum vacua with no method to pick out the right one. This fits well with the density operator formalism which, taken literally, suggests that the vacuum is not a part of physics but instead is simply an artifact of the mathematics.

And the large number of arbitrary constants do not appear so completely arbitrary. For example, experimental measurements of some of the neutrino mixing angles turn out to be small rational fractions of pi. And the masses of the leptons are related (to within experimental error) by the formula discovered by Yoshio Koide in 1982: $3(m_e+m_\mu+m_\tau) = 2(\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau})^2$, a 5−digit coincidence. As Dr. Koide writes[3], the presence of the square root "suggests that the charged lepton mass spectrum is not originated in the Yukawa coupling structure at the tree level, but it is given by a bilinear form on the basis of some mass generation mechanism." Density operators provide that bilinear form.

Any time a new formalism is found, it is natural to first pick the low hanging fruit. We will treat mass as if it were a force that converts left handed particles to right handed particles and vice versa. As a "force", mass is particularly simple because, in this model, the interactions correspond to Feynman diagrams with only two legs. For the density operator theory, these are the

low hanging fruit, and we will apply it to the Koide relation, extending it to the neutrinos. And we will finish with other, more speculative applications.

A word on the poetry that begins each chapter. These are works by a famous author. They were published sufficiently long ago that their copyright has expired. I quote them without attribution in the knowledge that, so long as our civilization survives, you will be able to quickly locate the author using the internet. Perhaps, after such a search, you will find that the unquoted portions of the poetry reads on the physics topic. And if you are insufficiently interested to make this small effort, a proper citation will provide you no advantage.

Regarding citations of other's work, this text is intended as a practical work, a training tool for graduate students more interested in the methods than the authors. A bit of stolen doggerel:

When 'Omer smote 'is bloomin' lyre,
    He'd 'eard men sing by land an' sea;
An' what he thought 'e might require,
    'E went an' took – the same as me!

Carl Brannen
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January 19, 2008

Introduction

TRY as he will, no man breaks wholly loose
    From his first love, no matter who she be.
Oh, was there ever sailor free to choose,
    That didn't settle somewhere near the sea?

The Thief's Lover

To be added later. At the moment I can't get myself to actually put on paper what I am thinking of for this, but it amounts to a short story.

Foundations

Lesser men feign greater goals,
    Failing whereof they may sit
Scholarly to judge the souls
    That go down into the pit,
And, despite its certain clay,
Heave a new world toward the day.

THIS TEXT DIFFERS FROM the usual introductions to elementary particles in that it assumes that elementary particles are best represented in density matrix form rather than state vector form. In quantum mechanics it is often said that density matrices are an alternative, equivalent, method of representing quantum states and this is true. Where density matrices differ from state vectors is in quantum field theory. Since the standard model of elementary particle physics is a quantum field theory, the difference between the theories gives a different persepctive on elementary particles. In this chapter we review the foundations of quantum mechanics with an eye to justifying the use of density matrices instead of state vectors.

0.1 Particle Waves

The first surprise that quantum mechanics brought to physics was the fact that matter can interfere with itself. Let a beam of particles impinge on a barrier with two slits. This produces two beams of spreading particles on the far (right) side of the barrier. These two beams are caught on a screen. One finds that the particles produce an interference pattern on the screen. See Fig. (0.1).

The intereference patterns are reminiscent of those resulting from classical waves, however, the image is formed by a large number of individual particle hits on the screen, and the image will form even when the experiment is run at such low particle production rates that only a single particle is present in the apparatus at a time.

The pattern only forms when a particle is allowed to go through either of the two slits. If, for example, one arranged for two different particles sources to feed the slots, one feeding only the upper slit and the other feeding only the lower

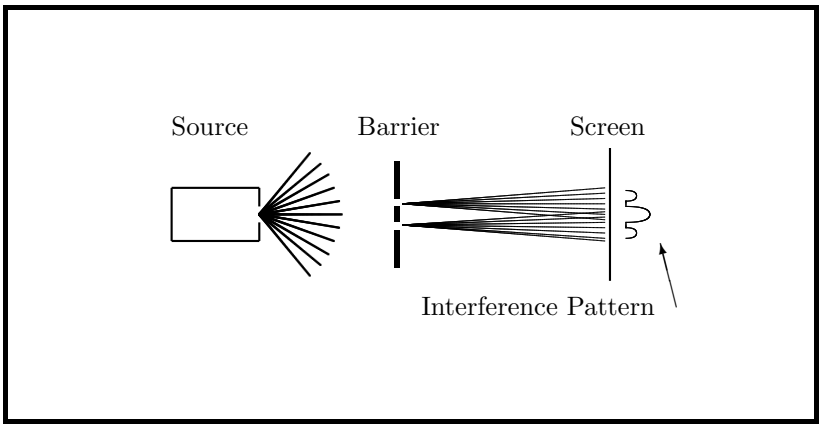


Figure 0.1: The Two Slit Experiment: Quantum particles impinge on a barrier with two slits. Particles that pass the barrier form an image on the screen that exhibits interference.

slit, the interference pattern disappears. The implication is that the particle somehow interferes with itself by passing through both slits simultaneously.

For any single particle, quantum mechanics assumes that it is impossible to predict exactly where the particle will end up. Instead, quantum mechanics allows one to compute a probability density. Given a wave function $\psi(x;t)$,[1] the probability density $\rho(x;t)$ is given by $\rho(x;t) = |\psi|^2 = \psi^*\psi$ where a^* indicates the complex conjugate of a.

In classical wave interference, one supposes that two wave sources, say A and B are present in the same region. These two wave sources produces two waves $F_A(x;t)$ and $F_B(x;t)$ which we will take, for simplicity to be real valued functions of position and time. In order for interfere to occur, F_A and F_B must be able to take both positive and negative values. The interference occurs because their signs cause them to partially cancel when they are added together to give the total wave, $F = F_A(x;t) + F_B(x;t)$.

In summing two classical (or quantum) waves, we are making the assumption of the "law of superposition". Classically, superposition tends to work for waves of sufficiently small amplitude. In quantum mechanics, superposition always works, but is of a nature somewhat different from classical superposition as will discuss in the next chapter.

Quantum mechanics is a probabilistic theory and the results of a calculation is a probability. Since probabilities cannot be negative, and yet matter waves must be able to interfere, one cannot take the quantum wave function to be a probability. Instead, one takes the squared magnitude, $|\psi|^2$, of the wave function as the probability. In the classical case, the squared magnitude of a

[1] We will use x to mean the x-coordinate or all the spatial coordinates, but we will try to carefully use the semicolon to separate the space and time parameters, for example, $\psi(x;t) = \psi(x,y,z;t)$.

wave function is the energy.

In a classical wave, both the amplitude and phase are observable. For example, one can measure the height and arrival times of waves on the ocean. With the right instruments, one can similarly measure the amplitude and phase of an electromagnetic waves. One can do this without significantly modifying the classical wave. One can therefore arrange for experiments with waves with known amplitude and phase.

The situation in quantum mechanics is more difficult. The squared amplitude of the probability wave gives the probability density for position $|\psi(x;t)|^2$, while the momentum operator $i\hbar\nabla$ applied to the wave function gives the wave function for momentum, ψ_p:

$$\psi_p(x;t) = i\hbar\nabla\psi(x;t). \tag{1}$$

The probability density for momentum is then $|\psi_p(x;t)|^2$.

If we multiply the wave function $\psi(x;t)$ by a complex phase $\exp(i\kappa)$ where κ is real, neither of the probability densities, that for momentum or position, is changed. More generally, let Q be an operator, for which we wish to calculate an average value. We have:

$$\langle Q_\psi\rangle(t) = \int \psi^*(x;t)Q\psi(x;t)\; d^3x, \tag{2}$$

where the subscript on Q denotes that this is the average for the quantum state ψ, which average in general depends on the time t. Define $\psi'(x;t) = \exp(i\kappa)\psi(x;t)$ with κ real so that ψ' is related to ψ by an overall (global) phase change. Since ψ contributes bilinearly in the formula for average value, Eq. (2), the average value for Q will be the same for ψ and ψ'.

We are left in the somewhat contradictory situation that the phase of a quantum state is apparently unneeded in physical measurements of the system it represents, but is necessary for interference to occur. The contradiction will be resolved by going to the density matrix representation.

0.2 Wave Function Density Matrices

In the usual state vector formalism, the fundamental object is the wave function $\psi(x;t)$. This must be modified to produce a probability density by multiplying by $\psi^*(x;t)$. However, if one so multiplies a wave function, one loses the phase information. A method that allows both the probability and phase information to be directly encoded in the same mathematical object is to instead multiply by $\psi^*(x';t)$ where x' is allowed to vary independently of x. The result is the "density matrix" wave function:

$$\rho(x,x';t) = \psi^*(x';t)\;\psi(x;t). \tag{3}$$

Since the density matrix wave function depends bilinearly on ψ, multiplying the state ψ by an arbitrary complex phase results in no change to the density

matrix wave function. The unphysical arbitrary complex phase has been removed. In addition, the probability density for position is now given by the "diagonal elements" of the density matrix:

$$|\psi(x;t)|^2 = \rho(x,x;t). \tag{4}$$

To get the probability density for momentum, we

0.3 Consistent Histories Interpretation

Consistent Histories section content goes here

Chapter 1

Finite Density Operators

For remember (this our children shall know: we are too near for that knowledge)
Not our mere astonied camps, but Council and Creed and College—
All the obese, unchallenged old things that stifle and overlie us—
Have felt the effects of the lesson we got—an advantage no money could buy us!

THE STANDARD PRACTICE in quantum mechanics has been to treat the state vector as the fundamental description of a quantum state and the density operator as a derived object. In this chapter we reverse this relationship and treat the density operator as the fundamental description of a quantum state.

1.1 Traditional Density Operators

In this section we introduce density operators as they are commonly taught, with a bit of an emphasis on the fundamental nature of them. We will loosely skim the excellent class notes of Frank C. Porter[4], to which the reader is directed.

We begin with a state space, with a countable orthonormal basis $\{|u_n\rangle, n = 1, 2, ...\}$. A system in a normalized state $|\psi(t)\rangle$ at time t can be expanded as:

$$|\psi(t)\rangle = \sum_n a_n(t)|u_n\rangle. \tag{1.1}$$

Normalization implies that $\sum_n |a_n(t)|^2 = 1$.

An observable Q can be expanded in this basis as:

$$Q_{mn} = \langle u_m|Q|u_n\rangle, \tag{1.2}$$

and the expectation value of $Q(t)$ for the system $|\psi(t)\rangle$ is:

$$\langle Q\rangle = \langle\psi(t)|Q\psi(t)\rangle = \sum_n \sum_m a_m^*(t)a_n(t)Q_{mn}. \tag{1.3}$$

Note that $\langle Q \rangle$ is quadratic in the coefficients a_n.

Define the density operator $\rho(t)$ as:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|. \tag{1.4}$$

Pure density operators are idempotent.

Since $|\psi(t)\rangle$ is normalized, the density operator is idempotent:

$$\rho^2(t) = \rho(t). \tag{1.5}$$

Writing $\rho(t)$ in the u_m basis we have:

$$\rho_{mn} = \langle u_m|\psi(t)\rangle\langle\psi(t)|u_n\rangle = a_m(t)a_n^*(t). \tag{1.6}$$

These matrix elements appear in Eq. (1.3) and consequently we can rewrite the expectation value of Q using the density operator:

$$\begin{array}{rcl} \langle Q \rangle(t) & = & \sum_m \sum_n a_m(t) a_n^*(t) Q_{mn} \\ & = & \mathrm{tr}(\rho(t)\, Q). \end{array} \tag{1.7}$$

Let $\{q\}$ be a subset of the eigenvalues of Q. Define $P_{\{q\}}$ as the projection operator that selects these eigenvalues. Then the probability that a measurement will lie in $\{q\}$ is

$$P(\{q\}) = \mathrm{tr}(P_{\{q\}}). \tag{1.8}$$

If $\{q\}$ is the whole spectrum of Q, then the projection operator is unity and the probability is one. Thus:

$$\mathrm{tr}(\rho(t)) = 1. \tag{1.9}$$

The time evolution of a state $|\psi(t)\rangle$ is given by Schroedinger's equation:

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{1.10}$$

where $H(t)$ is the Hamiltonian operator. When put into density operator form, the equation becomes:

$$\frac{d}{dt}\rho(t) = \frac{1}{i}[H(t), \rho(t)]. \tag{1.11}$$

Density operators are on an equal footing with state vectors in the foundations of quantum mechanics.

We have showed that the density operator $\rho(t)$ allows computation of expectation values and probabilities, and we've shown the equation for time evolution. This is apparently all that can be known about a quantum state, so the density operator is an alternative formulation for quantum mechanics on an equal footing with the state vector formalism from which we derived it.

1.2 An Alternative Foundation for QM

With two alternative formulations for quantum mechanics we have a choice. The two methods will give the same answer, but for any particular problem, one or the other is likely to give an easier calculation. And one or the other

might be closer to the underlying physics. We now look at the two formulations from the point of view of which is more likely to be useful in understanding the foundations of physics.

Both the density operator formalism and the state vector formalism share the same operators so there is no difference here. But the state vector formalism also requires states and in this sense the density operator formalism gets by with fewer mathematical objects. Since the operators alone are sufficient to describe a quantum state, the state vectors are only ancillary mathematical devices used for calculational convenience.

Density operator formalism is particularly well suited to statistical analysis of quantum mechanical systems. For example, the entropy of a quantum ensemble is defined by the simple equation:

$$S = -k \text{ tr}(\rho \ \ln(\rho)). \tag{1.12}$$

This is an advantage for the density operator formalism.

Given two solutions to the Schroedinger equation, $|\psi(t)\rangle$ and $|\phi(t)\rangle$, any linear combination is also a solution. That is, the solution set is linear. The same cannot be said of the density operator formulation. The advantage is with the state vector formalism, but this is a calculational advantage only, and later in this chapter we will show that linear superposition can be translated advantageously into the density operator formalism. Our models of reality are not inherently simpler when they are linear, instead they are simpler to use in calculations.

Calculations in the state vector formalism use an inner product which is inherently complex valued, while the corresponding calculations in the usual density operator formalism use the trace of a matrix. The trace is a complex function defined on the set of operators. In later chapters we will give a geometric interpretation of these complex numbers that will allow us to make calculations that are difficult or impossible in the state vector language.

Density operators have no phase ambiguity.

States represented by a state vector carry a phase ambiguity while the density operator states are completely defined. This is an advantage for the density operator formulation. We will later show that when translating a density operator state into state vector form, one must reintroduce this phase ambiguity in the form of a choice of spin direction. Thus the origin of the gauge symmetries appears to be related to a geometric choice.[1]

1.3 Eigenvectors and Eigenmatrices

Up to this point we've been discussing density operators in general. We will now specialize to the pure density operators, that is, the ones that correspond to state vectors. If the author mentions "density operator" or "density matrix"

[1] One might suppose that the density operator formulation would be at a disadvantage to the state vector form on problems associated with gauge forces, but this was recently shown not to be the case by Brown and Hiley.[5] Also see [6].

the reader should assume that he means "pure density operator". Furthermore, we will almost entirely be dealing with spin and internal degrees of freedom.

For the remainder of this chapter we will explore the density operator theory of the Pauli algebra. The usual introduction to the Pauli algebra involves the representation known as the Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.13}$$

Most physicists would associate the Pauli spin matrices with a representation of a Lie algebra, but in this book we will instead associate them with a Clifford algebra. The reader is not expected to know anything of Clifford algebra; we will introduce the necessary concepts in this chapter and the next. For now, let us only mention that Clifford algebras are also Lie algebras, but there are many Lie algebras that are not Clifford algebras.

In order to illustrate the uses of the pure density operators, we now discuss the classic eigenvector problem[7, §54] for spin-1/2 from the point of view of state vectors and density operators.

> A particle with spin 1/2 is in a state with a definite value $s_z = 1/2$. Determine the probabilities of the possible values of the components of spin along an axis z' at an angle θ to the z-axis.

This problem can be solved in a number of ways. Landau uses the fact that the mean spin vector of the particle along the z' directions is $\cos(\theta)$, along with the fact that this average is given by $(w_+ - w_-)/2$ where $w_\pm$ are the probabilities for the spin value along a' being measured as $\pm 1/2$. Then, since $w_+ + w_- = 1$, the result, $w_+ = (1 + \cos(\theta))/2$, can be deduced by algebra.

A more direct way of solving this problem is to find eigenvectors corresponding to spin-1/2 oriented in the z and z' directions, and then computing the probability with the formula $P = |\langle z|z'\rangle|^2$. This method is somewhat involved. If the vectors z and z' were more arbitrary, the problem would be even worse.

In the density operator formalism, the states are operators along with the particles. So the solution of the eigenvector equations are trivial. As with the state vector formalism, let us first define the operator for spin-1/2 in an arbitrary direction. To do this, we first define:

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \tag{1.14}$$

where σ_χ are the usual Pauli spin matrices. Then the projection operators for spin in the $\vec{u}$ direction is given by:

$$\vec{u} \cdot \vec{\sigma} = u_x \sigma_x + u_y \sigma_y + u_z \sigma_z. \tag{1.15}$$

where u_χ are the components of the vector $\vec{u}$.

The traditional way of solving this problem with state vectors is to write out the eigenvector equation with the operator for spin-1/2 in the $\vec{u}$ direction, which is simply half the projection operator:

$$((1/2)\vec{u}\cdot\vec{\sigma})\ \ |u+\rangle = (+1/2)\ \ |u+\rangle, \tag{1.16}$$

then substituting the Pauli spin matrices for $\vec{\sigma}$, and then solving the resulting matrix equation. This can be a fairly involved activity for the new student. In addition, even if the student uses the technique that requires normalization of the eigenvectors, their phases are still arbitrary.

One would think that solving the same problem with density operators would be more difficult because there are more apparent degrees of freedom with the state, but this is not the case. First, one must know how two projection operators multiply:

Multiplication of Pauli matrices.

$$(\vec{u}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) = \vec{u}\cdot\vec{v} + i(\vec{u}\times\vec{v})\cdot\vec{\sigma}. \tag{1.17}$$

In particular, when $\vec{u} = \vec{v}$ in the above equation, one finds that the right hand side is one.

Thus the matrix eigenvector equation is solved trivially by $(1+\vec{u}\cdot\vec{\sigma})$. That is:

Density operator equations have a simple closed form solution.

$$((1/2)\vec{u}\cdot\vec{\sigma})\ \ (1+\vec{u}\cdot\vec{\sigma}) = (+1/2)(1+\vec{u}\cdot\vec{\sigma}). \tag{1.18}$$

While this is a solution to the matrix eigenvector equation, it is not normalized as density operators are supposed to be, that is, $\rho^2 = \rho$ by Eq. (1.5). Instead, the square of $(1+\vec{u}\cdot\vec{\sigma})$ is twice itself, so our normalization is off by a factor of two. The correct density operator corresponding to spin-1/2 in the $\vec{u}$ direction is therefore given by:

$$\rho_u = (1+\vec{u}\cdot\vec{\sigma})/2, \tag{1.19}$$

and we have a closed form solution for the density operator eigenmatrix problem in an arbitrary direction $\vec{u}$.

When one tries to solve the general spin-1/2 eigenvector problem in the state vector formalism, one discovers that it is not so simple as the matrix problem. One does not have the easy option of writing the answer in terms of the Pauli matrices themselves (and therefore avoiding any mention of the particular representation chosen). One finds that ones solution fails for certain vectors, which we now illustrate.

Canceling out the factor of 1/2 on both sides, the state vector eigenvector equation is:

The corresponding state vector equation has no closed form solution.

$$\begin{pmatrix} u_z & u_x - iu_y \\ u_x + iu_y & -u_z \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.20}$$

or

$$\begin{pmatrix} u_z - 1 & u_x - iu_y \\ u_x + iu_y & -u_z - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{1.21}$$

An obvious solution to this problem is

$$\begin{pmatrix} u_x - iu_y \\ 1 - u_z \end{pmatrix}, \tag{1.22}$$

however, when $\vec{u} = (0, 0, 1)$, the above is zero and cannot be normalized. Another obvious solution is:

$$\begin{pmatrix} 1 + u_z \\ u_x + iu_y \end{pmatrix}, \tag{1.23}$$

but this solution is zero when $\vec{u} = (0, 0, -1)$. In addition to these two solutions, we can choose any linear combination. As the astute reader recognizes, any of these more general solutions will also be zero for some value of $\vec{u}$.

So we see that the problem, when written in terms of the pure density operators, has a simple general solution, but when written in the traditional state vector form, the problem is more difficult.

Density matrix formalism is more powerful.

Let us illustrate the power of the density operator formalism by restating the given problem in more general terms:

> A particle with spin 1/2 is in a state with a definite value $s_u = 1/2$ for spin measured in the direction $\vec{u}$. Determine the probability of a measurement of spin $+1/2$ along the $\vec{v}$ axis.

By symmetry, we know that the answer to the above problem is $(1 + \cos(\theta))/2$ where θ is the angle between $\vec{u}$ and $\vec{v}$. To solve it with the state vector formalism, we compute the eigenvectors for the two directions, then take the probability as the square of their dot product. In the density operator formalism, the answer is given by a trace:

$$w_+ = \mathrm{tr}(\rho_u \rho_v), \tag{1.24}$$

where ρ_χ is the density operator for spin in the $\vec{\chi}$ direction.

Using the formula for the density operator solution of the eigenvector problem, Eq. (1.19), the probability computes as:

$$\begin{aligned} w_+ &= \mathrm{tr}(\rho_u\ \rho_v) \\ &= \mathrm{tr}((1 + \vec{u} \cdot \vec{\sigma})/2(1 + \vec{v} \cdot \vec{\sigma})/2) \\ &= \mathrm{tr}(1 + (\vec{u} + \vec{v}) \cdot \vec{\sigma}/2 + (\vec{u} \cdot \vec{\sigma})(\vec{v} \cdot \vec{\sigma}))/4. \end{aligned} \tag{1.25}$$

The trace function keeps only the scalar part of its argument, and since the representation is done with 2×2 matrices, we have that $\mathrm{tr}(1) = 2$. Applying Eq. (1.17), we obtain:

$$w_+ = 2(1 + \vec{u} \cdot \vec{v})/4 = (1 + \cos(\theta))/2, \tag{1.26}$$

the same as with the state vector calculation.

1.4 Bras and Kets

In the previous section we saw that the density operator formulation allows the spin eigenvector problem to be solved in closed form while the state vector formulation cannot. Consequently, when the theory is taught in the state vector formulation, problems are left in eigenvector form rather than solved. The reason for the failure was that for any given general form solution in the state vector language, there is a direction where the solution becomes zero. But as we saw above, it is possible to find two complementary solutions, that is, solutions complementary in that one or the other (or both) provide solutions for any given direction. In this section we further discuss this fact in the context of how one obtains a state vector from a density operator.

Density operator formalism is naturally independent of the choice of representation of the Pauli algebra.

One of the strengths of the density operator formulation is that it allowed us to write the normalized solution to the eigenvector equation without reference to the representation of the Pauli algebra: $(1 + \vec{u} \cdot \vec{\sigma})/2$. But to show the connection to state vectors, let us write this solution explicitly using the Pauli matrices:

$$\begin{aligned} \rho_u &= (1 + \vec{u} \cdot \vec{\sigma})/2 \\ &= \tfrac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix}. \end{aligned} \tag{1.27}$$

Comparing the vectors of the above with Eq. (1.22) and Eq. (1.23), we see that the density operator eigenmatrix solution is composed of the two obvious solutions to the state vector eigenvector problem.

Now the two obvious state vector solutions to the eigenvector problem can be written in "square spinor" [8] form by filling in the unneeded columns of the matrices with zero. Then those two solutions sum to the density operator solution as follows:

$$\frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + u_z & 0 \\ u_x + iu_y & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & u_x - iu_y \\ 0 & 1 - u_z \end{pmatrix}. \tag{1.28}$$

In the above, the left hand side is the density operator solution, and the right hand side are two (not normalized) solutions to the state vector problem.

In the context of density operators, a natural method of converting the density operator solution to one of the square spinor solutions on the right hand side of Eq. (1.28) is to multiply by another density operator. In particular, note that the matrices for spin-1/2 in the $\pm z$ direction, that is, $\rho_{\pm z} = (1 \pm \sigma_z)/2$;

$$\begin{aligned} \rho_{+z} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \rho_{-z} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{1.29}$$

convert the density operator solution to the two complementary spinor solu-

tions:

$$\begin{array}{rcl} \frac{1}{2}\begin{pmatrix} 1+u_z & 0 \\ u_x+iu_y & 0 \end{pmatrix} & = & \frac{1}{2}\begin{pmatrix} 1+u_z & u_x-iu_y \\ u_x+iu_y & 1-u_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \frac{1}{2}\begin{pmatrix} 0 & u_x-iu_y \\ 0 & 1-u_z \end{pmatrix} & = & \frac{1}{2}\begin{pmatrix} 1+u_z & u_x-iu_y \\ u_x+iu_y & 1-u_z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \tag{1.30}$$

A column vector can be kept in matrix ("square spinor") form.

The matrices on the left hand side above are equivalent to spinors. To see this, notice that matrices that have all but one column zero act just like vectors. That is, the zeroes are conserved under multiplication by a constant, under addition with another square spinor matrix, and under multiplication by an arbitrary matrix on the left. These are all the things we require of kets.

Putting what we have found into density operator language, we find that the way one converts a density operator into ket form is to simply multiply on the right by a constant density operator. In the examples above, multiply on the right by $\rho_{\pm z}$.

Bras and kets defined using density operators.

For the moment let us choose the positive spin$-1/2$ in the $+z$ direction. The ket is defined as:

$$|u\rangle = \rho_u\ \rho_{+z} \tag{1.31}$$

If one reverses the order:

$$\langle v| = \rho_{+z}\ \rho_v \tag{1.32}$$

one obtains the bra form. These bras and kets are not normalized. Consequently, if we are to perform calculations, we must divide by the proper normalization constant, a subject we will take up after discussing scalars.

In the state vector formalism, multiplying a bra by a ket gives a scalar. Let's work out what it does in density operator formalism:

$$\begin{array}{rcl} \langle v|u\rangle & = & \rho_{+z}\ \rho_v\ \rho_u\ \rho_{+z} \\ & = & \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+u_z & u_x-iu_y \\ u_x+iu_y & 1-u_z \end{pmatrix}\begin{pmatrix} 1+v_z & v_x-iv_y \\ v_x+iv_y & 1-v_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ & = & \frac{1}{4}\begin{pmatrix} 1+u_z & u_x-iu_y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+v_z & 0 \\ v_x+iv_y & 0 \end{pmatrix} \\ & = & \frac{1}{4}(1+u_z)(1+v_z) + (u_x-iu_y)(v_x+iv_y)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{array} \tag{1.33}$$

Complex numbers as complex multiples of idempotents.

Note that the above is a complex multiple of ρ_{+z}. The set of all such matrices act just like the complex numbers. For example, if a and b are arbitrary complex numbers, then:

$$\begin{array}{rcll} a\rho_{+z} + b\rho_{+z} & = & (a+b)\rho_{+z}, & \text{and} \\ (a\rho_{+z})(b\rho_{+z}) & = & (ab)\rho_{+z}, & \end{array} \tag{1.34}$$

so complex multiples of ρ_{+z} act just like complex numbers. This means that if we define all of our bras and kets in a consistent manner with ρ_{+z}, our bras and kets will multiply to complex multiples of ρ_{+z}, and these act just like the complex numbers.

It remains to normalize the bras and kets we've defined in Eq. (1.32) and Eq. (1.31). Using the bra ket multiplication formula in Eq. (1.33), we have that

$$\langle u|u\rangle \quad = \quad \tfrac{1}{4}(1 + 2u_z + u_z^2 + u_x^2 + u_y^2). \tag{1.35}$$

So the normalization factor for $\langle v|u\rangle$ is the square root of the above multiplied by the same thing for v. That is, since probabilities are proportional to the squared magnitude of $\langle v|u\rangle$, the probability of the spin being measured as $+1/2$ in the v direction is:

$$w_+ = \frac{\langle v|u\rangle\langle u|v\rangle}{\langle v|v\rangle\langle u|u\rangle} \tag{1.36}$$

We leave it as an exercise for the reader to verify that the above reduces to $(1 + \vec{u}\cdot\vec{v})/2$.

We have shown that density operators can be converted to spinor form by pre or post multiplying by ρ_{+z}, a constant density operator. This brings the spinors into density operator form, but there is a problem. If $u = -\vec{z}$, then the product $\rho_{-z}\ \rho_{+z}$ is zero. This problem is identical to the issue that spinors had when we tried to write down a general solution to the eigenvector problem. We can get around it as we did before, by choosing a different vacuum density operator. Or better, by avoiding the spinor formalism where possible.

The "vacuum" density operator state is an arbitrary choice.

Since the z direction is not in any way special, our analysis of how to convert density operators to bras and kets using ρ_{+z} can be redone with any other constant density operator. The choice of this constant density operator defines the phase of the bra and ket that is produced. We will discuss this in greater detail in a later sections and chapters. For now, let us choose the notation ρ_0 to specify a constant density operator that need not be aligned in the $+z$ direction. For reasons that will become clear in later chapters, we will follow Julian Schwinger[9] and call ρ_0 the "vacuum" state.

Bras and kets from vacuum state.

So with ρ_0 as our choice of constant density operator, the conversion from density operators to bras and kets is:

$$\begin{array}{rcl} |u\rangle & = & \rho_u\ \rho_0 \\ \langle v| & = & \rho_0\ \rho_u. \end{array} \tag{1.37}$$

We now prove that these definitions give bra-kets that are complex multiples of ρ_0.

Write $\rho_0 = |0\rangle\langle 0|$, and similarly for u and v. Let M be an arbitrary operator. Then the matrix element of M for u and v is:

$$\begin{array}{rcl} \langle u|M|v\rangle & \rightarrow & \rho_0\ \rho_u\ M\ \rho_v\ \rho_0 \\ & = & |0\rangle\langle 0||u\rangle\langle u|M|v\rangle\langle v||0\rangle\langle 0| \\ & = & \langle 0|u\rangle\langle u|M|v\rangle\langle v|0\rangle|0\rangle\langle 0|, \\ & = & \langle 0|u\rangle\langle u|M|v\rangle\langle v|0\rangle\ \rho_0. \end{array} \tag{1.38}$$

which is seen to be a complex multiple of ρ_0. The result is that this definition differs from the usual only in the normalization. From the above, we see that the normalization can be fixed by dividing bras by $\langle 0|u\rangle$ and dividing kets by

$\langle v|0\rangle$. From this normalization it is clear why it is that this method only works when $|0\rangle$ is not antiparallel with any of the bras and kets which are being converted.

1.5 Naughty Spinor Behavior

For the Pauli algebra, we convert from density operator formalism to state vector formalism by choosing a vacuum, a constant pure density operator ρ_0. An arbitrary pure Pauli density operator has, at any given point in space-time, a spin orientation. This gives us a geometric interpretation of the source of the arbitrary complex phase seen in the state vector formalism. We can apply this interpretation to explain the odd behavior of spinors from a density operator point of view.

Spinors are naughty.

It is well known that when a spinor is rotated by 2π it does not return to its original value but instead is multiplied by -1. Since density operators do not have an arbitrary complex phase, this behavior cannot happen with the density operators.

The operator that rotates a spinor by an angle λ around a rotation axis defined by the vector $\vec{u}$ is simply:

$$U(\lambda) = e^{i\lambda\vec{u}\cdot\vec{\sigma}/2} = e^{i\lambda\sigma_u/2} \tag{1.39}$$

Let $|v\rangle$ be an arbitrary ket. We can write:

$$|v\rangle = (1+\sigma_u)/2\,|v\rangle + (1-\sigma_u)/2\,|v\rangle. \tag{1.40}$$

Applying the rotation operator to this gives:

$$\begin{aligned} U(\lambda)\,|v\rangle &= U(\lambda)(1+\sigma_u)/2\,|v\rangle + U(\lambda)(1-\sigma+u)/2\,|v\rangle \\ &= e^{+i\lambda/2}(1+\sigma_u)/2\,|v\rangle + e^{-i\lambda/2}(1-\sigma_u)/2\,|v\rangle, \end{aligned} \tag{1.41}$$

where we have taken advantage of the fact that $(1+\sigma_u)/2$ is both a projection operator and an eigenvector of σ_u. Putting $\lambda = 2\pi$ gives

$$\begin{aligned} U(2\pi)|v\rangle &= -(1+\sigma_u)/2\,|v\rangle - (1-\sigma_u)/2\,|v\rangle, \\ &= -|v\rangle. \end{aligned} \tag{1.42}$$

The above showed how one rotates a ket. To rotate a bra, one puts the rotation operator on the other side, and because of the complex conjugate, the spin operator takes a negative angle, $U(-\lambda)$

Density operators are well behaved.

If we replace $|v\rangle$ with a density operator, or any other operator, the same mathematics would apply. What is different about density operators is how they are rotated. For a density operator, the rotation operator must be applied to both sides of the density operator. This gives two factors of -1. Thus a density operator is unmodified when rotated through 2π using the rotation operators.

Applying the rotation operator to a spinor made from density operators, we see what the source of the factor of -1 is. When a spinor made from

density operators is to be rotated by spinors, one must include an extra rotation operator. Done the density operator way we have:

$$\begin{aligned} & U(\lambda)|u\rangle\langle u|U(-\lambda)\; U(\lambda)|0\rangle\langle 0|U(-\lambda) \\ = \;& U(\lambda)|u\rangle\langle u|0\rangle\langle 0|U(-\lambda). \end{aligned} \tag{1.43}$$

Putting $\lambda = 2\pi$ leaves the state unchanged, consistent with the fact that density operators are unchanged by rotations of 2π. Thus, from the density operator point of view, the -1 that a ket takes on rotation by 2π is a consequence of failing to rotate the vacuum bra, $\langle 0|$.

Density operators are simple in normalization.

In the state vector formulation of QM, there is a conflict between normalization and linearity.[2] The linear combination of two normalized state vectors is generally not a normalized state vector. If, on the other hand, we associate the states with the rays then we retain a sort of linearity, but our formula for probabilities becomes more complicated as we must normalize. Since density operators are essentially non linear, there is no temptation to sacrifice uniqueness for linear superposition.

1.6 Linear Superposition

As we saw in the previous section, the lack of arbitrary complex phase makes density operators a natural way of representing quantum states. On the other hand, an advantage of spinors is that they allow linear superposition. That is, given two spinors $|A\rangle$ and $|B\rangle$, and two complex numbers, a and b, we can define the linear superposition:

$$|aA + bB\rangle = a\,|A\rangle + b\,|B\rangle. \tag{1.44}$$

Linear superposition requires a choice of complex phase.

For any two given spinors, for example, $|A\rangle$ and $|B\rangle$, the linear superposition is well defined. But it is not stressed to those learning physics that the linear superposition is not well defined for the quantum states A and B. That is, to define $|aA + bB\rangle$, we must first choose kets to represent A and B. And since the choice of ket is arbitrary up to a complex phase, the linear superposition is also arbitrary.

If we do not require that the kets be normalized the arbitrariness of linear superposition becomes extreme. For example, let A be $+1/2$ spin in the $+z$ direction, and let B be $+1/2$ spin in the $-z$ direction. For the ket representing A and B, let u and v be arbitrary non zero real numbers. Then we can choose:

$$|+z\rangle \;=\; \begin{pmatrix} u/a \\ 0 \end{pmatrix} \quad , \quad |-z\rangle \;=\; \begin{pmatrix} 0 \\ v/b \end{pmatrix}, \tag{1.45}$$

and the linear superposition gives almost any quantum state:

$$a\,|+z\rangle + b\,|-z\rangle = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.46}$$

[2] In contrast to classical E&M, quantum mechanics, even in the usual state vector formulation, is not physically linear. Three times a state vector is a state vector that corresponds to the same physical situation (with the normalization changed), not a physical situation with three times as many particles or particles that are three times stronger.

This is the problem of linear superposition for spinors.

Quantum states are a part of physics, spinors are only mathematics.

The world is presumably composed of quantum states rather than spinors, so it we would like to have a method of defining linear superposition on quantum states rather than on spinors. But the above demonstrates that in making such a definition we must make some sort of choice. For spinors, the choice consists of the arbitrary complex phases of the two (or more) spinors which we wish to use, a rather inelegant definition.

In Eq. (1.31) we defined kets from the density operator formalism by choosing a "vacuum" state and multiplying on the right by this state. We can therefore define linear superposition by choosing a vacuum state, using it to convert the (unique) density operators to bras and kets:

$$\begin{aligned} |aA+bB\rangle &= a\ \rho_A\ \rho_0 + b\ \rho_B\ \rho_0, \\ \langle aA+bB|a &= a\ \rho_0\ \rho_A + b\ \rho_0\ \rho_B. \end{aligned} \tag{1.47}$$

Linear superposition for density operators requires a choice of vacuum state.

And we can now multiply our ket by our bra to get what we will show to be a complex multiple of a pure density operator corresponding to the linear superposition:

$$\begin{aligned} k\ \rho_{aA+bB,0} &= (a\ \rho_A\ \rho_0 + b\ \rho_B\ \rho_0,)(a\ \rho_0\ \rho_A + b\ \rho_0\ \rho_B) \\ &= (a\ \rho_A\ + b\ \rho_B,)\ \rho_0\ (a\ \rho_A + b\ \rho_B), \end{aligned} \tag{1.48}$$

where k is a complex constant (and may be zero). Note that choosing $a = 1, b = 0$ or $a = 0, b = 1$ gives a result that is proportional to ρ_A or ρ_B, respectively, as in the usual linear superposition.

Let X and Y be arbitrary operators, not necessarily pure states, and $\rho_0 = |0\rangle\langle 0|$ be a pure density operator. We now show that the product $X\ \rho_0\ Y$ is a complex multiple of a pure density operator. Compute the square:

$$\begin{aligned} (X\ \rho_0\ Y)^2 &= X\ \rho_0\ Y\ X\ \rho_0\ Y, \\ &= X\ |0\rangle\langle 0|\ Y\ X\ |0\rangle\langle 0|\ Y, \\ &= X\ |0\rangle\ \ (\langle 0|\ Y\ X\ |0\rangle)\ \ \langle 0|\ Y. \end{aligned} \tag{1.49}$$

The quantity in parentheses in the above is a complex number and so can be factored out of the operator product to give:

$$\begin{aligned} &= \langle 0|\ Y\ X\ |0\rangle\ (X\ |0\rangle\langle 0|\ Y), \\ &= \langle 0|\ Y\ X\ |0\rangle\ (X\ \rho_0\ Y). \end{aligned} \tag{1.50}$$

And since $X\ \rho_0\ Y$ squares to a complex multiple of itself, it is therefore an idempotent multiplied by that complex number. It remains to show that $X\ \rho_0\ Y$ is a primitive idempotent. To do this, compute the trace:

$$\mathrm{tr}(X\ \rho_0\ Y) = \mathrm{tr}(X\ |0\rangle\langle 0|\ Y) = \langle 0|\ Y\ X\ |0\rangle. \tag{1.51}$$

Since the trace is precisely the complex multiple of Eq. (1.50), $X\ \rho_0\ Y$ divided by this multiple is a pure density operator.

Chapter 2

Geometry

I have stated it plain, an' my argument's thus,
("It's all one," says the Sapper),
There's only one Corps which is perfect – that's us;
An' they call us Her Majesty's Engineers,
Her Majesty's Royal Engineers,
With the rank and pay of a Sapper!

THE PROGRAM OF contemporary physics is to produce a unified description of nature by looking for symmetries between the forces and particles. Where the forces are not symmetric, similarities are looked for and "spontaneous symmetry breaking" is assumed. While this plan has been successful in making great progress, the forward movement of that progress has been stalled for some years. The primary difficulty appear to be that symmetry principles allow too many different possibilities, and their application leaves too many arbitrary parameters that must be supplied by experiment.

This book will break with tradition and instead assume that geometry is at the foundation of physics. Our wave functions will be written in terms of scalars, vectors, pseudo vectors and pseudo scalars. These correspond to the traditional objects of geometry known to the ancients, points, lines, planes and volumes. The reader can suppose the correspondence between the traditional geometric objects and the geometry we will use here correspond to stresses induced by particles in the fabric of space-time, but this is not necessary, and we will not enlarge on the idea.

Traditional quantum mechanics is written with complex numbers. These are used in very particular ways in the standard theory. In this chapter we take advantage of these peculiarities and show that we can replace them with a geometric theory.

2.1 Complex Numbers

Let ρ_0 be any pure density operator, and let M be any operator. Then the product

$$\rho_0 \, M \, \rho_0 \tag{2.1}$$

is a complex multiple of ρ_0 as can be seen by replacing ρ_0 with the its spinor representation, $|0\rangle\langle 0|$. For example, choosing 0 to be spin+1/2 in the $+z$ direction, we have:

$$\begin{aligned} \rho_0 M \rho_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= M_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.2}$$

In the density matrix formalism, the complex numbers are a subset of the operators, and which subset depends on the choice of vacuum.

Any product of operators that begins and ends with the same pure density operator thus provides a version of the complex numbers. But it needs to be stressed that the form of these complex numbers depend on the choice of the vacuum operator.

Numbers, real or complex, are not a sort of operator.

Since our complex numbers depend on the choice of vacuum, we cannot follow the usual assumption of the spinor formalism which interprets the complex number $a + ib$ as a complex multiple of the unit matrix. That is, we will distinguish the complex numbers from the operators:

$$a + ib \neq \begin{pmatrix} a + ib & 0 \\ 0 & a + ib \end{pmatrix}. \tag{2.3}$$

For us, the complex numbers are only a mathematical convenience used for calculational purposes. Our complex numbers will arise from products that begin and end with the same pure density operator, and when we refer to them as complex numbers it is only for the convenience of not having to haul around the pure density operator that defines them. A logical consequence is that we should distinguish between the unity "1" of the complex numbers and the Pauli algebra. We will do this when convenient, but the old habits are hard to break.

Given this use of the complex numbers, we need to clear up the interpretation of the use of complex numbers in the definitions of the Pauli matrices. First, let us note that there are real representations of the Pauli algebra, for example:

$$\begin{pmatrix} 0 & & \\ & +1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 0 & \\ & & +1 \end{pmatrix}, \begin{pmatrix} +1 & & \\ & -1 & \\ & & 0 \end{pmatrix}. \tag{2.4}$$

Second, the imaginary unit of the Pauli matrices can be obtained from the product of the three Pauli matrices:

$$\sigma_x \sigma_y \sigma_z = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \tag{2.5}$$

and so avoid the use of complex multiples of operators. As an example of our use of complex numbers, consider the commutation relations that define the Pauli algebra. Rather than writing $[\sigma_x, \sigma_y] = 2i\sigma_z$, we will instead write:

$$[\sigma_x, \sigma_y] = 2(\sigma_x\sigma_y\sigma_z)\ \sigma_z = 2\sigma_x\sigma_y, \tag{2.6}$$

where the second equality comes from the fact that $\sigma_z\sigma_z = 1$.

The complex commutation relations of the Pauli algebra are replaced by a real Clifford algebra.

After replacing the imaginary unit in the commutation relations, we can break apart the commutator and turn the equations into somewhat simpler anticommutation relations. Along with the fact that the squares of the σ_χ square to unity, this gives a definition of the Pauli algebra that avoids complex operators:

$$\begin{aligned} \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \\ \sigma_x\sigma_y = -\sigma_y\sigma_x \\ \sigma_y\sigma_z = -\sigma_z\sigma_y \\ \sigma_z\sigma_x = -\sigma_x\sigma_z. \end{aligned} \tag{2.7}$$

These equations form the definition of a Clifford algebra. More complicated Clifford algebras will be the subject of later chapters and will be explained then.

Using the equations of Eq. (2.7), any product of Pauli algebra elements can be reduced in length to $\pm$ a product of at most three different Pauli algebra elements. For example:

$$\begin{aligned} \sigma_x\sigma_y\sigma_y\sigma_y\sigma_x\sigma_z\sigma_x &= \sigma_x(\sigma_y\sigma_y)\sigma_y\sigma_x\sigma_z\sigma_x = \sigma_x\hat{1}\sigma_y\sigma_x\sigma_z\sigma_x \\ &= \sigma_x(\sigma_y\sigma_x)\sigma_z\sigma_x = -\sigma_x(\sigma_x\sigma_y)\sigma_z\sigma_x \\ &= -(\sigma_x\sigma_x)\sigma_y\sigma_z\sigma_x = -\sigma_y\sigma_z\sigma_x. \end{aligned} \tag{2.8}$$

Furthermore, these products can be rearranged so that σ_x, σ_y and σ_z appear in that order. For example:

$$\begin{aligned} \sigma_z\sigma_y\sigma_x &= \sigma_z(\sigma_y\sigma_x) = -\sigma_z(\sigma_x\sigma_y) \\ &= -(\sigma_z\sigma_x)\sigma_y = +(\sigma_x\sigma_z)\sigma_y \\ &= \sigma_x(\sigma_z\sigma_y) = -\sigma_x(\sigma_y\sigma_z) \\ &= -\sigma_x\sigma_y\sigma_z. \end{aligned} \tag{2.9}$$

There are eight possible end results for these products, they are the "Pauli unit multivectors":

Pauli unit multivectors

$$\begin{array}{cccc} \hat{1}, & \sigma_y, & \sigma_x\sigma_y, & \sigma_x\sigma_z, \\ \sigma_x, & \sigma_z & \sigma_y\sigma_z, & \sigma_x\sigma_y\sigma_z. \end{array} \tag{2.10}$$

When operators are represented by 2×2 complex matrices, there are four complex numbers and therefore four complex degrees of freedom. Since we are avoiding the use of complex numbers as operators, it makes sense to think of the 2×2 complex matrices as having eight real degrees of freedom. Those degrees of freedom are given by the Pauli unit multivectors. But the Pauli unit multivectors are written in geometric form, that is, they are written in terms of x, y and z. Thus the Pauli unit multivectors give us a geometric interpretation of the 2×2 complex matrices. For example,

Operators always have a geometric interpretation.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv (\hat{1} + \sigma_z)/2$$
$$\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \equiv (\sigma_x\sigma_y\sigma_z + \sigma_x\sigma_y)/2 \tag{2.11}$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv (\sigma_x - \sigma_x\sigma_z)/2$$

Since any product of Pauli matrices is $\pm$ one of the Pauli unit multivectors, the Pauli unit multivectors are sufficient to represent any operator that is made from Pauli matrices. That is, real linear combinations of the Pauli unit multivectors are closed under multiplication.

The Pauli blades are: scalars, vectors, pseudovector and pseudoscalar.

The eight Pauli unit multivectors can be divided into subsets in several useful ways. Clearly $\hat{1}$ is a scalar. Following the tradition from Clifford algebra, σ_x, σ_y, and σ_z are called vectors, $\sigma_y\sigma_z$, $\sigma_z\sigma_x$ and $\sigma_x\sigma_y$ are "pseudovector", and $\sigma_x\sigma_y\sigma_z$ is a "pseudoscalar". These are the four "blades" that we mention here only for completeness. Blade value is conserved under addition but is not conserved under multiplication.

Signatures are always +1 or -1

The Pauli unit multivectors give either $+1$ or -1 when squared. This is called the signature, and we can divide the eight elements according to signature:

$$\begin{aligned} \hat{1}^2 &= \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = +1 \\ (\sigma_x\sigma_y\sigma_z)^2 &= (\sigma_y\sigma_z)^2 = (\sigma_z\sigma_x)^2 = (\sigma_x\sigma_y)^2 = -1. \end{aligned} \tag{2.12}$$

In the Pauli algebra, the signature can be thought of as the presence or absence of the imaginary unit $\sigma_x\sigma_y\sigma_z$. In later chapters we will be working with more complicated Clifford algebras whose elements have more interesting signatures. It turns out that the idempotents of a Clifford algebra are defined by the elements that square to $+1$. Signature is conserved under addition, but not conserved under multiplication.

Orientations are n, x, y, and z.

Finally, the Pauli unit multivectors can be organized according to orientation. Three of the four orientations are the x, y and z directions. The fourth is n, the neutral direction. Orientation is preserved under addition and under multiplication follows these rules:

			$\times$	n	x	y	z
$\hat{1}$	$\sigma_x\sigma_y\sigma_z$	$\in$	n	n	x	y	z
$\hat{x}$	$\sigma_y\sigma_z$	$\in$	x	x	n	z	y
$\hat{y}$	$\sigma_z\sigma_x$	$\in$	y	y	z	n	x
$\hat{z}$	$\sigma_x\sigma_y$	$\in$	z	z	y	x	n

(2.13)

As the above table shows, under multiplication, orientation forms a finite group. For the Pauli algebra, the neutral orientation is equivalent to half of the trace. That is, if we use $n(M)$ to denote the extraction of the neutral portion of an

operator,

$$\begin{aligned}
\mathrm{tr}(M) &= \mathrm{tr}(M_1\hat{1} + M_x\sigma_x + M_y\sigma_y + M_z\sigma_z \\
&\quad + M_i\sigma_x\sigma_y\sigma_z + M_{ix}\sigma_y\sigma_z + M_{iy}\sigma_z\sigma_x + M_{iz}\sigma_x\sigma_y) \\
&= \mathrm{tr}\begin{pmatrix} M_1 + M_z + iM_i + iM_{iz} & M_x - iM_y + iM_{ix} + M_{iy} \\ M_x + iM_y + iM_{ix} - M_{iy} & M_1 - M_z + iM_i - iM_{iz} \end{pmatrix} \\
&= 2(M_1 + iM_i) \\
&= 2\, n(M).
\end{aligned} \tag{2.14}$$

We have some use of orientation in this chapter, and later on it will be useful in classifying the primitive idempotents of more complicated Clifford algebras.

The three Pauli matrices σ_x, σ_y, and σ_z are associated with the three dimensions x, y and z. Special relativity requires a time dimension, t, and to model this spacetime requires that we add a sort of σ_t to the three Pauli algebras. Each time one adds a new dimension to a Clifford algebra, the number of unit multivectors is multiplied by two, so this will give us 16 unit multivectors. We will cover this algebra, the "Dirac algebra" and its "Dirac unit multivectors" in the next chapter and later chapters will add one more (hidden) dimension.

2.2 Expectation Values

There are at least three ways of defining expectation values of operators in the pure density operator formalism. At the risk of increasing confusion, we will consider all three in this subsection.

Given an operator M and a state A, in the usual bra-ket notation, one finds the expectation value of M for the state A by finding a normalized spinor for A, and then computing

$$\langle M\rangle_A = \langle A|M|A\rangle. \tag{2.15}$$

Vacuum expectation value defined.

To convert the above into an expectation value computed in the density operator formalism, we replace the spinors with the products of the pure density operator and the vacuum operator as explained in Eq. (1.37). The result is what we will call the "vacuum expectation value":

$$\langle M\rangle_{A,0} = \rho_0\ \rho_A\ M\ \rho_A\ \rho_0, \tag{2.16}$$

which, since it begins and ends with the vacuum operator ρ_0, we can interpret as a complex number. Writing Eq. (2.16) in spinor form:

$$\begin{aligned}
\langle M\rangle_{A,0} &= |0\rangle\langle 0|A\rangle\langle A|M|A\rangle\langle A|0\rangle\langle 0|, \\
&= \langle 0|A\rangle\ \langle A|0\rangle\ \langle A|M|A\rangle\ |0\rangle\langle 0|,
\end{aligned} \tag{2.17}$$

shows that our definition of expectation value differs from the usual spinor definition of Eq. (1.7) due to the influence of the vacuum. Of particular interest is the vacuum expectation value of the unit operator $\hat{1}$:

$$\begin{aligned}
\langle M\rangle_{1,0} &= \rho_0\ \rho_A\ \hat{1}\ \rho_A\ \rho_0, \\
&= \rho_0\ \rho_A\ \rho_0 \\
&= \langle 0|A\rangle\ \langle A|0\rangle\ |0\langle\rangle 0|.
\end{aligned} \tag{2.18}$$

From this we see that we obtain the usual expectation values by taking the ratio:

$$\langle M\rangle_A = \langle M\rangle_{A,0}/\langle\hat{1}\rangle_{A,0}, \tag{2.19}$$

where the ratio is to be interpreted in our loose use of complex numbers. For example, choosing ρ_0 to be spin $+1/2$ in the $+z$ direction,

$$a/b \equiv \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \Big/ \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.20}$$

Expectation value without vacuum.

We can also eliminate the vacuum from Eq. (2.16) and define the "expectation value" as:

$$\begin{aligned} \langle M\rangle_A &= \rho_A\, M\, \rho_A, \\ &= |A\rangle\langle A|M|A\rangle\langle A| \\ &= \langle A|M|A\rangle\ |A\rangle\langle A|, \end{aligned} \tag{2.21}$$

which defines the expectation value in terms of the ratio of $\rho_A\rho_M\rho_A$ to ρ_A. This is the form of expectation value that we will use most often, but we will use it only in the context of comparing expectation values for different operators M with respect to the same state A. If we wish to compare expectation values for two different states, we will have to choose a vacuum and use the earlier method of Eq. (2.16).

The third method of defining expectation values is to follow the spinor tradition and use the trace. For the Pauli algebra, there is a geometric imaginary unit, $\sigma_x\sigma_y\sigma_z$, which squares to -1 and commutes with all the elements of the algebra. But not all Clifford algebras have an imaginary unit and for such algebras defining the trace is more difficult. Accordingly, we will avoid the use of the trace.

For the expectation value to be real, we must place the same restriction on M as in the spinor theory, that is, M must be Hermitian. Note that as written, the expectation value depends on the choice of vacuum. In general, we will be concerned with operators that are not Hermitian and which therefore do not have real expectation values. For these operators, we can use the complex interpretation given above, or alternatively we can write the complex numbers in terms of $\sigma_x\sigma_y\sigma_z$.[1] For example, let the state be spin$+1/2$ in the $+z$ direction, and the operator be $M = 3 - 2\sigma_x\sigma_y + \sigma_x$. We can compute the expectation value several ways: In the spinor representation, one converts the operator into 2×2 complex matrices by using the Pauli matrices, finds the bra and ket associated with spin $+1/2$ in the $+z$ direction, and multiplies them together:

$$\begin{aligned} \langle M\rangle_{+z} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 3-2i & 1 \\ 1 & 3-2i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 3 - 2i \end{aligned} \tag{2.22}$$

In the density operator formalism, the same calculation can be done without use of the representation. The student will find it useful to work the example

[1] In later chapters we will generalize this appropriately.

out in detail. It can be done easily if one uses the anticommutation relations to move the left pure density operator ρ_{z+} over to the right side. Doing this will cancel out some parts of the operator and leave other parts unchanged:

$$\begin{aligned}
\langle M\rangle_{+z} &= (1+\sigma_z)/2\,(3-2\sigma_x\sigma_y+\sigma_x)\,(1+\sigma_z)/2 \\
&= (3-2\sigma_x\sigma_y+\sigma_x)\,(1+\sigma_z)/4 \\
&\quad +\sigma_z(3-2\sigma_x\sigma_y+\sigma_x)\,(1+\sigma_z)/4 \\
&= (3-2\sigma_x\sigma_y+\sigma_x)\,(1+\sigma_z)/4 \\
&\quad +(3-2\sigma_x\sigma_y-\sigma_x)\,\sigma_z\,(1+\sigma_z)/4 \\
&= (3-2\sigma_x\sigma_y+\sigma_x)\,(1+\sigma_z)/4 \\
&\quad +(3-2\sigma_x\sigma_y-\sigma_x)\,(1+\sigma_z)/4 \\
&= (3-2\sigma_x\sigma_y)\,(1+\sigma_z)/2, \\
&= (3-2\sigma_x\sigma_y\sigma_z)\,(1+\sigma_z)/2.
\end{aligned} \tag{2.23}$$

The last equality was obtained by noting that $(1+\sigma_z)/2 = \sigma_z(1+\sigma_z)/2$, that is, σ_z is an eigenvector of ρ_{z+} with eigenvalue 1 so we can introduce factors of it; that is, $\sigma_z(1+\sigma_z) = (1+\sigma_z)$.

In the above calculation, the 3 and $2\sigma_x\sigma_y\sigma_z$ components contributed to the expectation value while the σ_x component does not. Let us write M_χ as the operator that is any one of the eight degrees of freedom of the Pauli algebra. That is, $M_1 = \hat{1}$, $M_x = \sigma_x$, ..., $M_{iz} = \sigma_x\sigma_y$, $M_i = \sigma_x\sigma_y\sigma_z$. In computing $\langle M_\chi\rangle$, we should note that for any choice of χ, M_χ will either commute or anticommute with ρ_{+z}. And of course M_χ commutes with $\hat{1}$. This allows us to compute the expectation value easily. If M_χ anticommutes with σ_z, then the expectation is zero:

$$\begin{aligned}
\langle M\rangle_{+z} &= (0.5(1+\sigma_z)M_\chi)0.5(1+\sigma_z) \\
&= (M_\chi 0.5(1-\sigma_z))0.5(1+\sigma_z) \\
&= 0
\end{aligned} \tag{2.24}$$

because $(1+\sigma_z)$ and $(1-\sigma_z)$ annihilate each other. On the other hand, if M_χ commutes, then the expectation is nonzero:

$$\begin{aligned}
\langle M\rangle_{+z} &= (0.5(1+\sigma_z)M_\chi)0.5(1+\sigma_z) \\
&= (M_\chi 0.5(1+\sigma_z))0.5(1+\sigma_z) \\
&= M_\chi 0.5(1+\sigma_z),
\end{aligned} \tag{2.25}$$

because $0.5(1+\sigma_z)$ is idempotent.

This method of calculation is very useful and it is worth describing again why it works. If an idempotent is written with unit multivectors, one can always factor a unit multivector to the other side of the idempotent. This will either leave the idempotent unaltered, or it will change the idempotent to a different idempotent that will annihilate the original idempotent. This behavior will be repeated when we later study the Dirac and more complicated Clifford algebras.

To calculate with primitive idempotents, factor unit multivectors around the primitive idempotents, which sometimes changes the primitive idempotents.

Given the idempotent $0.5(1+\sigma_z)$, the Pauli unit multivectors that commute with it, and therefore give a nonzero expectation value, are $\hat{1}$, σ_z, $\sigma_x\sigma_y$ and

$\sigma_x\sigma_y\sigma_z$. The other four Pauli unit multivectors will convert ρ_{+z} to ρ_{-z} which will annihilate with the other ρ_{+z} and therefore give an expectation value of zero. Returning to the classification of the degrees of freedom of the Pauli unit multivectors according to orientation, we see that the neutral and z oriented unit multivectors give nonzero expectation values while the x and y oriented unit multivectors give zero expectation values. Thus half the degrees of freedom give a zero expectation value and the other half gives a nonzero expectation. When we later study more complicated Clifford algebras we will find somewhat more complicated behavior; in addition to different orientations giving zero expectation, there is also the possibility of changing the "internal" quantum states which will then annihilate.

Hermitian Pauli operators include scalars and vectors, but not pseudoscalar and pseudovector.

A Hermitian operator can be written as a sum of real multiples of the scalar and vector blades of the Pauli algebra. That is, the general Hermitian operator H can be written as:

$$H = H_1\hat{1} + H_x\sigma_x + H_y\sigma_y + H_z\sigma_z, \tag{2.26}$$

where H_χ are real numbers. An anti-Hermitian operator can be written similarly as a sum of real multiples of psuedovectors and the psuedoscalar blades.

In the Pauli algebra, the pure density operators are of the form:

$$\rho_u = (1 + \sigma_u)/2 = (1 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z)/2, \tag{2.27}$$

where (u_x, u_y, u_z) is a unit vector, and are therefore Hermitian. Consequently, when we compute the expectation value of a pure density operator,

$$\langle\rho_u\rangle_A = \rho_A\ \rho_u\ \rho_A, \tag{2.28}$$

the result will be a real number (i.e. a real multiple of ρ_A).

Transition probabilities and the probability postulate.

When a system is prepared in a state A, and we later measure it to see if it is in a state B, it is a fundamental postulate of quantum mechanics that the "transition probability" will be given by the expectation value of ρ_B. That is,

$$\begin{aligned} P(A \to B) &= |\langle A|B\rangle|^2, \\ &= \langle A|B\rangle\langle B|A\rangle, \\ &= \langle A|\rho_B|A\rangle. \end{aligned} \tag{2.29}$$

Transition probabilities are a very important type of expectation value and we will be computing them often in this book.

2.3 Amplitudes and Feynman Diagram

In this section we will analyze a particularly simple sort of Feynman diagram, one which begins and ends with states that can be defined in the Pauli algebra. While these are very limited in type, they will see extensive use later in this book in the context of the origin of mass. We will begin with the state I represented by the pure density operator ρ_I, and end with the state F represented by the pure density operator ρ_F.

Feynman devotes a section titled "Path Integral Formulation of the Density Matrix" in his lecture notes [10] on statistical mechanics. We differ in our presentation in that we are dealing with the internal degrees of freedom (i.e. spin), while Feynman was dealing with the external degrees of freedom (i.e. position or momentum). But the expansions are otherwise similar.

In the method of Feynman diagrams, one computes the probability of a transition from an initial state I to a final state F by first computing the "amplitude" of the transition as a sum over various paths, and then computing the probability as the square of the absolute value of the amplitude. In the case of the simple Feynman diagrams we are working with here we know that the amplitude is simply $\langle I|F\rangle$ which we will write as:

$$Amp(I \to F) = \langle I|F\rangle \quad \text{(spinor formalism)}. \tag{2.30}$$

Feynman amplitudes are not to be complex numbers.

In converting this to the density operator formalism, we have several choices. We could define geometric complex numbers by computing in the multiples of the I, F or vacuum 0 primitive idempotents ρ_I, ρ_F or ρ_0. However, Feynman amplitudes require only the operations of addition and computation of a magnitude, they do not require complex multiplication.

Let us consider what happens when we insert a new state between the initial and final states in the Feynman method. Since the Feynman method requires us to sum over all possible paths, we must also include the complement of the state. We can write the interior parts of this calculation in pure density matrix form. For example, if A is the state with spin $+1/2$ in the a direction:

$$\begin{aligned} Amp(I \to F) &= \langle I|F\rangle, \\ &= \langle I|\hat{1}|F\rangle, \\ &= \langle I|0.5(1+\sigma_a)|F\rangle + \langle I|0.5(1-\sigma_a)|F\rangle, \\ &= \langle I||A\rangle\langle A||F\rangle + \langle I||\bar{A}\rangle\langle\bar{A}||F\rangle, \\ &= \langle I||A\rangle\langle A||F\rangle + \langle I||\bar{A}\rangle\langle\bar{A}||F\rangle, \\ &= \langle I|\rho_A|F\rangle + \langle I|\bar{\rho_A}|F\rangle, \end{aligned} \tag{2.31}$$

where we have introduced the notation $\bar{\rho_A}$ to indicate the complement of the pure density operator ρ_A.

This idea can be continued to give a sequence of pure density operators between the initial bra and final ket. The various intermediate states are all treated as pure density operators, only the initial and final states are in ket form. This suggests that the natural way of writing an amplitude in pure density operator form is to convert the initial and final objects to look the same as the intermediate ones. In this manner, again inserting the state A and its complement between initial and final state we have:

Definition of amplitude in density operator formalism.

$$\begin{aligned} Amp(I \to F) &= \rho_I\ \rho_F, \\ &= \rho_I\ \rho_A\ \rho_F + \rho_I\ \bar{\rho_A}\ \rho_F. \end{aligned} \tag{2.32}$$

Using this formalism will allow us to connect Feynman diagrams together without having to distinguish between internal and external lines. But to justify

this, we need to show that we can add these sorts of objects, and that we can compute squared magnitudes of them.

For definitiveness, let I be spin $+1/2$ in the $+z$ direction, and let F be spin $+1/2$ in the $+x$ direction. With Pauli spin matrices, the various objects are:

$$\begin{aligned} \rho_I &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \rho_F &= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \\ \langle I| &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ |F\rangle &= \sqrt{0.5}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.33}$$

Suppose that we are considering a particular sequence of states that happens to reduce to the matrix $M = M_{ij}$. For the spinor method, we get an amplitude of:

$$\begin{aligned} Amp(I \to M \to F) &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\sqrt{0.5}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ &= \sqrt{0.5}(M_{11} + M_{21}). \end{aligned} \tag{2.34}$$

Replacing the spinors with pure density operators gives:

$$\begin{aligned} Amp(I \to M \to F) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \\ &= 0.5(M_{11} + M_{21})\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.35}$$

Amplitudes are, in general, products of primitive idempotents.

Instead of getting a complex number, the amplitude is a more general operator, a product of two primitive idempotents. Comparing Eq. (2.35) to Eq. (2.34), we see that summation of amplitudes will give analogous results. For example, if to the above amplitude we add another amplitude that has an internal operator of N_{ij}, then the two methods will give results that are proportional to $(M_{11} + M_{21}) + (N_{11} + N_{21})$. Therefore, to get a pure density operator formalism for amplitudes, we need only define the equivalent of the squared magnitude.

To convert the amplitude into a squared magnitude, we can use the methods of Sec. (2.2), that is, work with vacuum expectation values and use multiples of ρ_0, or work in expectation values using multiples of either ρ_I or ρ_F. For example, let us compute the squared magnitude for the amplitude of Eq. (2.35). In the traditional spinor formalism, the answer is the squared magnitude of Eq. (2.34), that is $0.5|M_{11} + M_{21}|^2$. We begin by multiplying the amplitude by its complex conjugate transpose:

$$\begin{aligned} P(I \to F) &= 0.5(M_{11} + M_{21})\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}0.5(M_{11}^* + M_{21}^*)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ &= 0.25|M_{11} + M_{21}|^2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= 0.5|M_{11} + M_{21}|^2\rho_F. \end{aligned} \tag{2.36}$$

If we multiply on the left by the complex conjugate the result is a multiple of the initial primitive idempotent:

$$\begin{aligned} P(I \to F) &= 0.5(M_{11}^* + M_{21}^*) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ 0.5(M_{11} + M_{21}) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} &= 0.25|M_{11} + M_{21}|^2 \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0.5|M_{11} + M_{21}|^2 \rho_I. \end{aligned} \tag{2.37}$$

We can also define the squared magnitude using an arbitrarily chosen vacuum state.

As before, these methods allow us to compute amplitudes without the need for specifying a representation of the Pauli algebra. In the above, we have used the Pauli spin matrices because of their familiarity. Let us now redo the calculation in geometric form. The amplitude is:

$$Amp(I \to M \to F) = 0.5(1 + \sigma_z)M0.5(1 + \sigma_x). \tag{2.38}$$

We will compute the squared magnitude in terms of the initial idempotent, $0.5(1 + \sigma_z)$. Then the squared magnitude is:

$$P(I \to M \to F) = 0.5(1 + \sigma_z)M0.5(1 + \sigma_x)0.5(1 + \sigma_x)M(1 + \sigma_z). \tag{2.39}$$

To compute this, we need to commute M around the idempotents. Therefore, we divide M into its orientations and write $M = M_n + M_x + M_y + M_z$ and compute:

$$\begin{aligned} P(I \to M \to F) &= 0.5(1 + \sigma_z)(M_n + M_x + M_y + M_z)0.5(1 + \sigma_x) \\ &\quad 0.5(1 + \sigma_x)(M_n^* + M_x^* + M_y^* + M_z^*)0.5(1 + \sigma_z). \end{aligned} \tag{2.40}$$

In our previous analysis of expectation values, we found that the x and y orientations give zero expectation values. Therefore, to perform the above computation we can multiply out the terms between $0.5(1+\sigma_z)$ and group terms according to their orientations. Only the "n" and "z" orientations survive giving:

$$\begin{aligned} & 0.5(M_n + M_z + M_x + M_y)(1 + \sigma_x)(M_n^* + M_x^* + M_y^* + M_z^*) \\ = \;& 0.5(M_n M_n^* + M_x M_x^* + M_y M_y^* + M_z M_z^*) \;\; \text{“}n\text{”} \\ + \;& 0.5(M_n \sigma_x M_x^* + M_x \sigma_x M_n^* + M_y \sigma_x M_z^* + M_z \sigma_x M_y^*) \;\; \text{“}n\text{”} \\ + \;& 0.5(M_n M_z^* + M_z M_n^* + M_y M_x^* + M_z M_n^*) \;\; \text{“}z\text{”} \\ + \;& 0.5(M_n \sigma_x M_y^* + M_x \sigma_x M_z^* + M_y \sigma_x M_n^* + M_z \sigma_x M_x^*) \;\; \text{“}z\text{”}. \end{aligned} \tag{2.41}$$

Since $\sigma_z(1 + \sigma_z) = (1 + \sigma_z)$, we can turn the "$z$" terms into "$n$" form by bringing in an extra factor of σ_z on the right:

$$\begin{aligned} = \;& 0.5(M_n M_n^* + M_x M_x^* + M_y M_y^* + M_z M_z^*) \\ + \;& 0.5(M_n \sigma_x M_x^* + M_x \sigma_x M_n^* + M_y \sigma_x M_z^* + M_z \sigma_x M_y^*) \\ + \;& 0.5(M_n M_z^* + M_z M_n^* + M_y M_x^* + M_x M_y^*)\sigma_z \\ + \;& 0.5(M_n \sigma_x M_y^* + M_x \sigma_x M_z^* + M_y \sigma_x M_n^* + M_z \sigma_x M_x^*)\sigma_z. \end{aligned} \tag{2.42}$$

The above is all in the "n" orientation; the only degrees of freedom are $\hat{1}$ and $\sigma_x\sigma_y\sigma_z$. Writing $M_n = m_n$, $M_x = m_x\sigma_x$, $M_y = m_y\sigma_y$, and $M_z = m_z\sigma_z$ allows us to simplify. In computing the result, we need to keep in mind the anticommutation relations so that, for instance, $\sigma_x\sigma_z\sigma_y = -\sigma_x\sigma_y\sigma_z$. The result is:

$$\begin{aligned} &= 0.5(m_n m_n^* + m_x m_x^* + m_y m_y^* + m_z m_z^* + m_n m_x^* + m_x m_n^* - i m_y m_z^* \\ &\quad + i m_z m_y^* + m_n m_z^* + m_z m_n^* - i m_y m_x^* + i m_x m_y^* + i m_n m_y^* + m_x m_z^* \\ &\quad - i m_y m_n^* + m_z m_x^*) \\ &= 0.5(m_n + m_x - i m_y + m_z)(m_n^* + m_x^* + i m_y^* + m_z^*) \\ &= 0.5|m_n + m_x - i m_y + m_z|^2 \end{aligned} \tag{2.43}$$

where "i" indicates multiplication by $\sigma_x\sigma_y\sigma_z$.

The use of $\sigma_x\sigma_y\sigma_z$ as i in the above calculation suggests that we should use it in defining complex multiples of elements of the Pauli algebra. For example, let I be the initial state and F the final state in a Feynman. We insert the state A between I and F as follows:

$$\rho_F\rho_A\rho_I, \tag{2.44}$$

and using the methods of this section we can treat these as complex numbers to the extent that we can add them together and compute squared magnitudes. But we can go a little farther than this, and we can write this object as a complex multiple of $\rho_F\rho_I$ as follows:

$$\rho_F\rho_A\rho_I = (a_R + a_I\sigma_x\sigma_y\sigma_z)\rho_F\rho_I, \tag{2.45}$$

where a_R and a_I are real numbers.

Amplitudes are complex numbers times the product of two primitive idempotents.

This is easiest to show using the Pauli spin matrices. Let us assume that $\rho_F = 0.5(1+\sigma_z)$ and that $\rho_I = 0.5(1+\cos(\theta)\sigma_z + \sin(\theta)\sigma_x)$ so that the angle between the spin axes of I and F is θ. Let M be an arbitrary 2×2 matrix. Then

$$\begin{aligned} \rho_F M \rho_I &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} 0.5 \begin{pmatrix} 1+\cos(\theta) & \sin(\theta) \\ \sin(\theta) & 1-\cos(\theta) \end{pmatrix}, \\ &= 0.5 \begin{pmatrix} m_{11} & m_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+\cos(\theta) & \sin(\theta) \\ \sin(\theta) & 1-\cos(\theta) \end{pmatrix}, \\ &= 0.5 m_{11} \begin{pmatrix} 1+\cos(\theta) & \sin(\theta) \\ 0 & 0 \end{pmatrix} + 0.5 m_{12} \begin{pmatrix} \sin(\theta) & 1-\cos(\theta) \\ 0 & 0 \end{pmatrix} \\ &= 0.5(m_{11} + m_{12}(1-\cos(\theta))/\sin(\theta)) \begin{pmatrix} 1+\cos(\theta) & \sin(\theta) \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.46}$$

since m_{11} and m_{12} are complex, the result is as desired. In the state vector formalism, amplitudes are also complex numbers, but they depend on the phases chosen for the spinors. In the density operator formalism, there is no arbitrary complex phase and the complex phase of the amplitude defined here is not arbitrary.

Complex phases of density operator amplitudes are uniquely defined.

2.4 Products of Density Operators

The calculation of Feynman diagram amplitudes using products of density operators as discussed in the previous section is the primary subject of this book. In this section, we derive some useful methods of reducing these sorts of calculations.

The first thing to note is that if we wish to expand an amplitude by inserting states between and initial and final state, there is no reason to do this if the initial and final states annihilate. Such a transition has zero probability and is not of interest. So we will assume that the initial and final states do not annihilate.

In the Pauli algebra, the pure density operator states correspond to primitive idempotents in various directions. Given three unit vectors, $\vec{u}$, $\vec{v}$, and $\vec{w}$, the corresponding pure density operator states are:

$$\begin{aligned} \rho_u &= 0.5(1+\sigma_u) = 0.5(1 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z), \\ \rho_v &= 0.5(1+\sigma_v), \\ \rho_w &= 0.5(1+\sigma_w). \end{aligned} \tag{2.47}$$

We will interpret the ρ_w as the initial state, ρ_u as the final state, and ρ_v as the intermediate state that is inserted between them in a Feynman amplitude calculation. Since antiparallel projection operators annihilate, under this assumption, we suppose that $\vec{u}$ and $\vec{w}$ are not antiparallel.

In the previous section, we showed that the product of ρ_u, ρ_v and ρ_w can be written as a "complex" multiple of the product of ρ_u and ρ_w. Let us write this complex number as a magnitude and phase in the following fashion:

$$\rho_u\rho_v\rho_w = \sqrt{R_{uvw}\exp(iS_{uvw})}\;\rho_u\rho_w, \tag{2.48}$$

where R_{uvw} and S_{uvw} are real functions of the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$. The factors of two are included for later convenience. Rewriting the above in spinor form we have:

$$\begin{aligned} \rho_u\rho_v\rho_w &= |u\rangle\langle u|v\rangle\langle v|w\rangle\langle w|, \\ &= \langle u|v\rangle\langle v|w\rangle\;|u\rangle\langle w|, \\ &= (\langle u|v\rangle\langle v|w\rangle/\langle u|w\rangle)\;|u\rangle\langle u|w\rangle\langle w|, \end{aligned} \tag{2.49}$$

and therefore

$$\sqrt{R_{uvw}\exp(iS_{uvw})} = \langle u|v\rangle\langle v|w\rangle/\langle u|w\rangle. \tag{2.50}$$

In the above, the left hand side is completely defined within the density matrix formalism and therefore does not depend on choice of phase. The right hand side is therefore invariant with respect to the arbitrary complex phases chosen for the spinors; while the individual amplitudes depend on the choice of phase, the above product does not.

Eq. (2.50) includes three spinor amplitudes; a typical one is $\langle u|v\rangle$. The squared magnitude of each of these can be written as $0.5(1+\cos(\theta_{uv}))$ where

θ_{uv} is the angle between $\vec{u}$ and $\vec{v}$. Similarly for the other three amplitudes, and this gives the formula for R_{uvw} as:

$$R_{uvw} = 0.5(1 + \cos(\theta_{uv}))(1 + \cos(\theta_{vw}))/(1 + \cos(\theta_{uw})). \tag{2.51}$$

It remains to find the equation for S_{uvw}.

The three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are on the surface of the unit sphere and define a spherical triangle, which we will call $\triangle uvw$. We next show that S_{uvw} is proportional to the oriented[2] area of that spherical triangle. To see this, pick a vector $\vec{x}$ inside the triangle $\triangle uvw$ that is not opposite to any of the three vectors (which would result in an inconvenient zero amplitude). Since $\langle w|x\rangle\langle x|w\rangle = |\langle w|x\rangle|^2$ is real, we can multiply by it without changing the phase. Accordingly, we can write S_{uvw} as:

$$\begin{aligned} S_{uvw} &= 2\arg(\langle u|v\rangle\langle v|\rangle/\langle u|w\rangle), \\ &= \arg((\langle u|x\rangle\langle x|w\rangle/\langle u|w\rangle)(\langle u|v\rangle\langle v|x\rangle/\langle u|x\rangle)(\langle v|w\rangle\langle w|x\rangle/\langle v|x\rangle)), \\ &= S_{uxw} + S_{uvx} + S_{vwx}, \end{aligned} \tag{2.52}$$

where in the last equality we have used that $\arg(\alpha\beta) = \arg(\alpha) + \arg(\beta)$. It remains to find the constant of proportionality. Reversing two points on an oriented spherical triangle defines a new spherical triangle that is complementary to the old one. The area of the triangle and its complement must add to the total area of the unit sphere, 4π, but the triangle and its complement must carry opposite phases. That is, since S_{uvw} is defined as twice the complex phase,

$$\begin{aligned} 0.5\ S_{uvw} &= -0.5\ S_{uwv} + 2n\pi, \\ 0.5\ S_{uvw} + 0.5\ S_{uwv} &= 2n\pi, \\ \text{Area}(\triangle uvw) + \text{Area}(\triangle uwv) &= 4\pi, \end{aligned} \tag{2.53}$$

where n is an integer, and the simplest solution is the one claimed:

$$S_{uvw} = \text{Area}(\triangle uvw), \tag{2.54}$$

the verification of which we leave as an exercise for the reader.

In order to solidify this method of reducing products of projection operators, we will now compute several products using Eq. (2.51) and Eq. (2.52). Let ρ_x, ρ_y, and ρ_z be spin $+1/2$ in the x, y, and z directions. Then

$$\rho_x = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \rho_y = \begin{pmatrix} 0.5 & -0.5i \\ +0.5i & 0.5 \end{pmatrix}, \rho_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.55}$$

For this example, $\theta_{xy} = \theta_{xz} = \theta_{yz} = \pi/2$ so the $0.5(1 + \cos(\theta))$ factors will each give 0.5, and so $R_{xyz} = 0.5(0.5)^2/0.5 = 0.25$. The spherical triangle is an octant, and so its area, $4\pi/8 = \pi/2$ is S_{xyz}. Multiplying out the matrices gives:

$$\begin{aligned} \rho_x\rho_y\rho_z &= \tfrac{1+i}{2}\begin{pmatrix} 0.5 & 0 \\ 0.5 & 0 \end{pmatrix}, \\ &= \sqrt{0.5e^{i\pi/2}}\rho_x\rho_z, \end{aligned} \tag{2.56}$$

[2]Oriented here means that order matters, for example, $S_{uwv} = -S_{uvw}$.

as claimed. Redoing the same problem in the geometric notation, $\rho_x = 0.5(1 + \sigma_x)$, etc., gives:

$$\begin{aligned}
\rho_x\rho_y\rho_z &= (0.5)^3(1+\sigma_x)(1+\sigma_y)(1+\sigma_z) \\
&= 0.125((1+\sigma_x)(1+\sigma_z) + (1+\sigma_x)\sigma_y(1+\sigma_z)) \\
&= 0.125(1+\sigma_x)(1+\sigma_z) \\
&\quad +0.125(1+\sigma_x)\sigma_x\sigma_y\sigma_z(1+\sigma_z)) \\
&= 0.5(1+\sigma_x\sigma_y\sigma_z)0.5(1+\sigma_x)0.5(1+\sigma_z) \\
&= 0.5(1+i)\rho_x\rho_z,
\end{aligned} \tag{2.57}$$

gives the same answer without need to choose a representation.

Products of more than three projection operators can be reduced using the above formula, provided one avoids a division by zero in Eq. (2.51), the definition of R_{uvw}. Earlier we saw that products of three projection operators that begin and end with the same projection operator are always real multiples of that same projection operator. Products of four projection operators are, in general, complex:

$$\begin{aligned}
\rho_u\rho_v\rho_w\rho_u &= \sqrt{R_{uvw}\exp(iS_{uvw})}\,\rho_u\rho_w\rho_u, \\
&= \sqrt{R_{uvw}R_{uwu}\exp(iS_{uvw})\exp(iS_{uwu})}\rho_u\rho_u, \\
&= \sqrt{R_{uvw}R_{uwu}\exp(iS_{uvw})}\rho_u,
\end{aligned} \tag{2.58}$$

and the phase, $S_{uvw}/2$, is the same as for the product of the first three projection operators, $\rho_u\rho_v\rho_w$. In addition, the product $R_{uvw}R_{uwu}$ simplifies:

$$\begin{aligned}
R_{uvw}R_{uwu} &= 0.5\frac{(1+\cos(\theta_{uv}))(1+\cos(\theta_{vw}))}{/}(1+\cos(\theta_{uw})) \\
&\quad \times 0.5\frac{(1+\cos(\theta_{uw}))(1+\cos(\theta_{wu}))}{/}(1+\cos(\theta_{uu})) \\
&= 0.5(1+\cos(\theta_{uv}))0.5(1+\cos(\theta_{vw}))0.5(1+\cos(\theta_{wu})),
\end{aligned} \tag{2.59}$$

to the product of three $0.5(1 + \cos(\theta))$ factors, as expected. This product is unchanged when u, v, and w are pairwise swapped. On the other hand, since the area is oriented, swapping any two of u, v, and w negates S_{uvw}:

$$S_{uvw} = S_{vwu} = S_{wuv} = -S_{uwv} = -S_{vuw} = -S_{wvu}. \tag{2.60}$$

If we interpret the product $\rho_u\rho_v\rho_w\rho_u$ as a Feynman amplitude, then physically, the path that the particle takes is to visit the states in the order u, w, v, and then back to u. When we swap v and w, we reverse the sequence of states. As we saw above, this causes the phase to be negated. In other words, the amplitude of the reversed sequence is the complex conjugate of the amplitude of the forward sequence. It is left as an exercise for the reader to show that this is generally true for more complicated paths. Taking the continuous limit, we have that paths that begin and end on the same point take a complex phase equal to half the area of the region encircled (possibly circled multiple times).

One last comment on products of four projection operators. In the spinor formulation, each of the legs corresponds to a complex number, and the final amplitude is the product of the three complex numbers. As noted before,

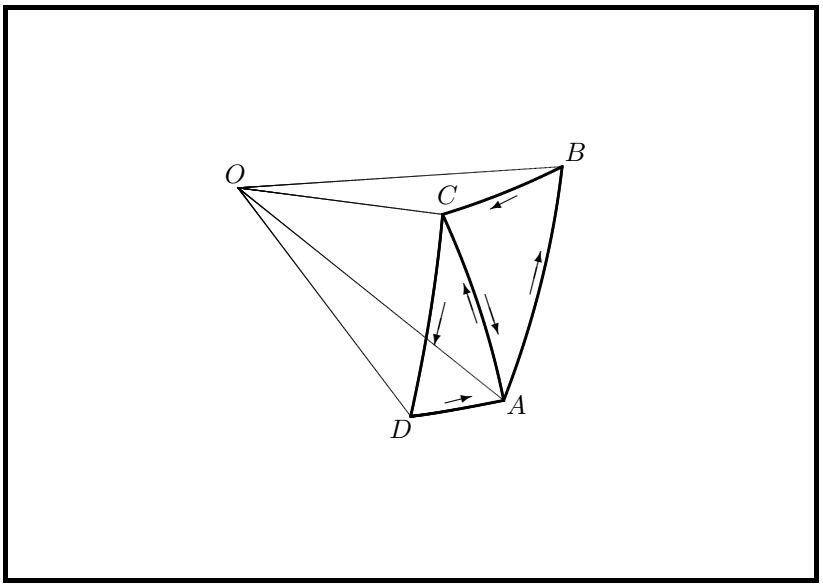


Figure 2.1: Two adjoining spherical triangles on the unit sphere. S_{uvw} is additive, that is, $S_{ABCD} = S_{ABC} + S_{ABD}$, and therefore the complex phase of a series of projection operators can be determined by the spherical area they encompass.

the actual complex phase for each leg will depend on the choices of arbitrary complex phases for the spinors, but the overall product will not depend on these choices.

Within the spinor formalism, it makes sense to attribute a phase contribution to each leg, that is, $S_{uvw} = S_{uv} + S_{vw} + S_{wu}$. This is reminiscent of Eq. (2.52) which gives S_{uvw} in terms of the three phases S_{uxw}, S_{uvx}, S_{vwx}. To make the equivalence more exact, we can choose the point x to be the vacuum, that is the constant projection operator ρ_0. Each of the three sides of the spherical triangle gives a contribution to S_{uvw}, that is,

$$S_{uvw} = S_{u0w} + S_{uv0} + S_{vw0}. \tag{2.61}$$

Since $S_{u0w} = -S_{uw0} = +S_{wu0}$, we can bring the vacuum to the right most position in the indices and the result is a symmetric form for the complex phase:

$$S_{uvw} = S_{wu0} + S_{uv0} + S_{vw0}. \tag{2.62}$$

The fact that $S_{uw0} = -S_{wu0}$ suggests that we consider the legs of the spherical triangle as ordered paths. That is, they can be traversed in either of two directions and the complex phase they contribute will take a sign depending on which direction is taken. This gives a natural way of interpreting the fact that S_{uvw} is proportional to the area. See Fig. (2.1). Consider a spherical

quadrilateral, $ABCD$. We can split the region into two spherical triangles by adding line AC. Letting the two triangles be traversed in the same direction starting at A, the left AC is traversed in opposite directions by the two triangles. Therefore the contribution to the complex phase by the leg AC is canceled and the complex phase of $ABCD$ is the sum of the complex phases of the two triangles.

Chapter 3

Primitive Idempotents

> Then I stripped them, scalp from skull, and my hunting dogs fed full,
> And their teeth I threaded neatly on a thong;
> And I wiped my mouth and said, "It is well that they are dead,
> For I know my work is right and theirs was wrong."
>
> But my Totem saw the shame; from his ridgepole shrine he came,
> And he told me in a vision of the night: –
> "There are nine and sixty ways of constructing tribal lays,
> And every single one of them is right!"

THIS CHAPTER WILL DISCUSS primitive idempotents for Clifford algebras in more depth than the Pauli algebra can provide. We will also cover idempotents in general, and "roots of unity", the elements that square to $\hat{1}$. Of particular interest are groups of idempotents that commute. In quantum mechanics, these correspond to measurements that can be made in any order, and are therefore "compatible". Also of interest are groups of commuting roots of unity; these we will use to define the quantum numbers of states.

3.1 The Pauli Algebra Idempotents

An "idempotent" is an element of an algebra that is unchanged when squared:

$$\rho^2 = \rho. \tag{3.1}$$

The above is a simple equation, but it is a nonlinear equation, and as such it is not necessarily very easy to solve.

A "primitive idempotent" is a nonzero idempotent which cannot be written as the sum of two nonzero idempotents. This definition can be confusing to the student. It seems circular. Let us apply it to several simple algebras and see how to understand it on an intuitive and physical level.

Complex numbers have only one primitive idempotent: 1.

For the complex numbers, the solutions to Eq. (3.1), that is, the idempo-

tents, are 0 and 1. Of these, only 1 is nonzero. The only possible sum of two nonzero idempotents is $1+1=2$, so 1 is a primitive idempotent of the complex (or real) numbers.

For the complex $2x2$ matrices, the idempotents are obtained by solving:

$$\begin{array}{rcl} \begin{pmatrix} a & c \\ b & d \end{pmatrix} & = & \begin{pmatrix} a & c \\ b & d \end{pmatrix}^2 \\ & = & \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix} \end{array} \tag{3.2}$$

There are four resulting equations:

$$\begin{array}{rcl} a & = & a^2+bc, \\ d & = & d^2+bc, \\ b & = & b(a+d), \\ c & = & c(a+d). \end{array} \tag{3.3}$$

If $a+d \neq 1$, then, by the last two of the above, $b=c=0$. The other two equations then give $a^2=a$ and $d^2=d$. There are four solutions of this sort, according as a and d are independently 0 or 1. Two of these have $a+d=1$, the other two are:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

the 0 and $\hat{1}$ of the algebra.

The solutions with $a+d=1$, are more interesting – they are simply the primitive idempotents of the Pauli algebra – but they are more difficult to find. To solve these equations, let us rewrite a, b, c, and d in geometric form:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a_1+a_z & a_x-ia_y \\ a_x+ia_y & a_1-a_z \end{pmatrix}, \tag{3.5}$$

where a_1, a_x, a_y, and a_z are complex numbers. Then $a+d=(a_1+a_z)+(a_1-a_z)=2a_1=1$, so $a_1=0.5$.

In solving $\rho^2=\rho$ for the case $a+d=1$, we can use either matrix arithmetic and square the right hand side of Eq. (3.5), or we can do the calculation directly in geometric notation:

$$\begin{array}{rcl} 0.5+a_x\sigma_x+a_y\sigma_y+a_z\sigma_z & = & (0.5+a_x\sigma_x+a_y\sigma_y+a_z\sigma_z)^2 \\ & = & (0.25+a_x^2+a_y^2+a_z^2) \\ & & +a_x\sigma_x+a_y\sigma_y+a_z\sigma_z, \end{array} \tag{3.6}$$

where the remaining terms cancel by anticommutation. The solution is $a_x^2+a_y^2+a_z^2=0.25$, and factoring 0.5 out gives the general solution as:

$$\rho = 0.5(\hat{1}+u_x\sigma_x+u_y\sigma_y+u_z\sigma_z), \tag{3.7}$$

where $u_x^2+u_y^2+u_z^2=1$. The full set of nonzero idempotents are the above, plus $\hat{1}$. These elements have scalar parts of 0.5 or 1. If we add any two of

these idempotents together, the sum will have a scalar part of at least 1, so the elements with scalar part 0.5, that is, the idempotents given in Eq. (3.7), are all primitive idempotents. Since $\hat{1} = 0.5(1 + \sigma_z) + 0.5(1 - \sigma_z)$, we have that $\hat{1}$ is not primitive.

If u_x, u_y, u_z are all real, then the primitive idempotent defined in Eq. (3.7) is one of the pure density operators of the Pauli algebra discussed in the first chapter. If one or more of them are complex, then we still have a primitive idempotent, but it is no longer Hermitian. For example, $u_x = 2.4i$, $u_y = 0$, $u_z = 2.6$ gives the following primitive idempotent:

Pure density operators of spin−1/2 are Hermitian primitive idempotents.

$$0.5 + 1.2i\sigma_x + 1.3\sigma_z = \begin{pmatrix} 1.8 & 1.2i \\ 1.2i & -0.8 \end{pmatrix} \tag{3.8}$$

This completes the solution of the primitive idempotents of the 2×2 complex matrices, and also the Pauli algebra. It should be clear that the geometric notation is very useful in solving the system of equations generated by $\rho^2 = \rho$.

3.2 Commuting Roots of Unity

All of the primitive idempotents of the Pauli algebra have the same scalar part, 0.5; they differ in the remaining part. This variable part squares to one:

$$\begin{array}{rcl} \iota_u & = & u_x\sigma_x + u_y\sigma_y + u_z\sigma_z, \\ \iota_u^2 & = & u_x^2 + u_y^2 + u_z^2 = \hat{1}. \end{array} \tag{3.9}$$

We will call elements of a Clifford algebra or matrix algebra that square to $\hat{1}$, "roots of unity". That they are square roots is assumed.

The equations can be reversed. Suppose that ι is not a scalar and that $\iota^2 = \hat{1}$. Then $0.5(1 \pm \iota)$ is an idempotent:

$$\begin{array}{rcl} (0.5(\hat{1} \pm \iota))^2 & = & 0.25(\hat{1} \pm 2\iota + \iota^2) \\ & = & 0.25(\hat{1} \pm 2\iota + \hat{1}) \\ & = & 0.5(\hat{1} \pm \iota). \end{array} \tag{3.10}$$

In the first chapter, we associated a product of two different primitive idempotents of the Pauli algebra with the complex numbers of a Feynman amplitude. We saw that these products do not commute, that is, $\rho_a\rho_b \neq \rho_b\rho_a$. This is a general property of *primitive* idempotents in matrix or Clifford algebras, that is, they either are identical, annihilate, or they don't commute.

But the products of distinct idempotents that are neither primitive nor equal to $\hat{1}$, even in matrix and Clifford algebras, need not be zero. For example, the following two 3×3 matrices are idempotent and commute, but their product is neither zero nor either of the two:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

Note that the product is also idempotent. This is true in general: if ρ_1 and ρ_2 commute and are idempotent, then $\rho_1\rho_2$ is also idempotent and commutes with ρ_1 and ρ_2:

$$\begin{aligned}(\rho_1\rho_2)^2 &= = \rho_1^2\,\rho_2^2 = \rho_1\rho_2,\\ \rho_1(\rho_1\rho_2) &= (\rho_1\rho_2)\rho_1,\\ \rho_2(\rho_1\rho_2) &= (\rho_1\rho_2)\rho_2.\end{aligned} \tag{3.12}$$

Products of commuting idempotents commute.

This fact generalizes. A set of commuting idempotents generate a group under multiplication that consists entirely of commuting idempotents.

Let $\rho_1 = 0.5(1+\iota_1)$ and $\rho_2 = 0.5(1+\iota_2)$ be two commuting idempotents. Then ι_1 and ι_2 commute:

$$\begin{aligned}\rho_1\,\rho_2 &= \rho_2\,\rho_1,\\ 0.5(1+\iota_1)0.5(1+\iota_2) &= 0.5(1+\iota_2)0.5(1+\iota_1),\\ 1+\iota_1+\iota_2+\iota_1\iota_2 &= 1+\iota_1+\iota_2+\iota_2\iota_1,\\ \iota_1\iota_2 &= \iota_2\iota_1.\end{aligned} \tag{3.13}$$

The above equations can be reversed: If ι_1 and ι_2 commute and square to $+1$, then $\rho_1 = 0.5(1+\iota_1)$ and $\rho_2 = 0.5(1+\iota_2)$ are commuting idempotents. Thus the sets of commuting roots of unity match the sets of commuting idempotents.

An idempotent is primitive only when it cannot be written as the sum of two other non zero idempotents. Let's suppose that an idempotent ρ is not primitive so we can write:

$$\begin{aligned}\rho &= \rho_1+\rho_2,\\ \rho^2 &= \rho_1^2+\rho_2^2+\rho_1\rho_2+\rho_2\rho_1,\\ \rho &= \rho_1+\rho_2+\rho_1\rho_2+\rho_2\rho_1, \text{ and so}\\ 0 &= \rho_1\rho_2+\rho_2\rho_1.\end{aligned} \tag{3.14}$$

This says that ρ_1 and ρ_2 anticommute. Let us reduce the product $\rho_2\rho_1\rho_2$ using anticommutation two different ways:

$$\begin{aligned}\rho_2\rho_1\rho_2 &= (\rho_2\rho_1)\rho_2 = -\rho_1\rho_2\rho_2 = -\rho_1\rho_2,\\ &= \rho_2(\rho_1\rho_2) = -\rho_2\rho_2\rho_1 = -\rho_2\rho_1.\end{aligned} \tag{3.15}$$

To prove an idempotent is not primitive, look for it among the sums of the idempotents that commute with it.

So $-\rho_2\rho_1\rho_2 = \rho_1\rho_2 = \rho_2\rho_1$, and we have that ρ_1 and ρ_2 commute. But since they commute with each other, they also must commute with their sum, ρ. This suggests that a way of finding primitive idempotents is to look at sets of commuting idempotents and eliminate all the ones that can be written as sums of the others. This is a finite problem.

We can make this problem into a "lattice". Given a set of commuting idempotents, define an inequality as follows:

$$\rho_L < \rho_G \text{ iff } (\rho_G - \rho_L) \text{ is a non zero idempotent.} \tag{3.16}$$

Equality is defined as usual. This definition is a "partial ordering", which means that it acts like the usual inequality does on the real numbers except that some elements cannot be compared.

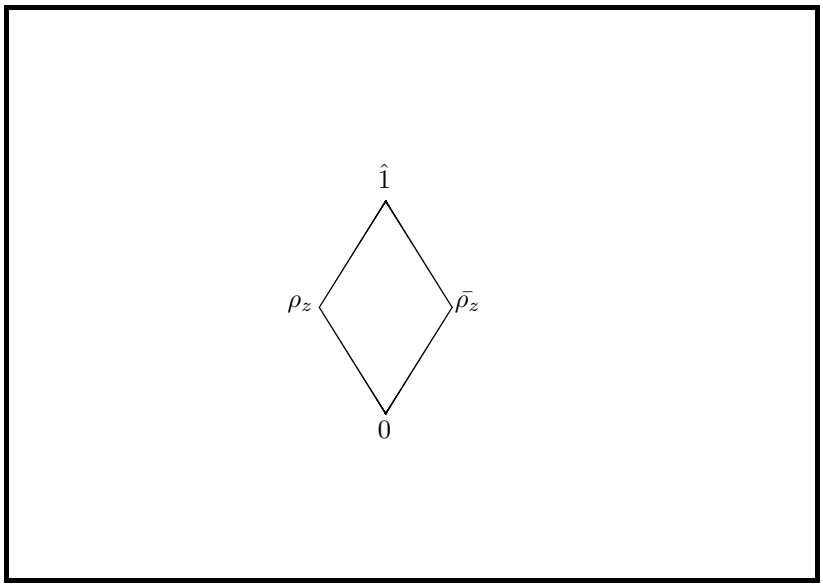


Figure 3.1: Drawing of the lattice of a complete set of commuting idempotents of the Pauli algebra, see text, Eq. (3.17).

As an example of this partial ordering, consider the set of four commuting idempotents of the Pauli algebra, 0, $\rho_z = 0.5(1 + \sigma_z)$, $\bar{\rho}_z = 0.5(1 - \sigma_z)$, and $\hat{1}$. These elements satisfy the following relationships:

	0	ρ_z	$\bar{\rho}_z$	$\hat{1}$
0	$=$	$<$	$<$	$<$
ρ_z	$>$	$=$		$<$
$\bar{\rho}_z$	$>$		$=$	$<$
$\hat{1}$	$>$	$>$	$>$	$=$

(3.17)

Any two elements can be compared except for ρ_z and $\bar{\rho}_z$. To make this into a lattice, put $\hat{1}$ at the top, 0 at the bottom, and the rest of the elements in between. If $\rho_L < \rho_G$, then arrange ρ_L to be closer to 0 than ρ_G. Add a line between two elements if they are ordered, and there is no other element between them. See Fig. (3.1). In general, if there is a relation between two elements, then there is a way of climbing from the lower one to the upper one. The primitive idempotents are the elements closest to the bottom. In quantum mechanics theory, two measurements are compatible (i.e. commute) if their idempotents are related by an inequality. For this reason, the lattice is sometimes called the "lattice of propositions" in the literature.

3.3 3×3 Matrices

We now consider the problem of the primitive idempotents of 3×3 matrices. As before, we will make these from commuting roots of unity. The diagonal matrices commute, so let's consider the diagonal roots of unity:

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.18}$$

that is,

$$\begin{aligned} a_1 &= \pm 1, \\ a_2 &= \pm 1, \\ a_3 &= \pm 1. \end{aligned} \tag{3.19}$$

The ± 1 in the above three equations are independent; there are thus $2^3 = 8$ roots of unity. When one takes all three right hand sides as $+1$, the root of unity is just $\hat{1}$. We can label these roots by a suffix that gives the signs. The eight roots are then:

$$\begin{aligned} \iota_{---} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \iota_{--+} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \\ \ldots &\quad \ldots \\ \iota_{+++} &= \hat{1}. \end{aligned} \tag{3.20}$$

Each of these eight roots of unity defines an idempotent (which may or may not be primitive) by $\rho_\chi = 0.5(1+\iota_\chi)$. The $+1$s on the diagonal of ι_χ produce $+1$s on the diagonal of ρ_χ, but the -1s become 0s. There are two idempotents that are particularly simple: $\rho_{+++} = \hat{1}$ and $\rho_{---} = 0$. The remaining six idempotents have mixtures of 0s and 1s on their diagonals in various combinations.

The partial ordering of the diagonal idempotents is as follows:

	0	ρ_{--+}	ρ_{-+-}	ρ_{+--}	ρ_{++-}	ρ_{+-+}	ρ_{-++}	$\hat{1}$
$0 = \rho_{---}$	$=$	$<$	$<$	$<$	$<$	$<$	$<$	$<$
ρ_{--+}	$>$	$=$				$<$	$<$	$<$
ρ_{-+-}	$>$		$=$		$<$		$<$	$<$
ρ_{+--}	$>$			$=$	$<$	$<$		$<$
ρ_{++-}	$>$		$>$	$>$	$=$			$<$
ρ_{+-+}	$>$	$>$		$>$		$=$		$<$
ρ_{-++}	$>$	$>$	$>$				$=$	$<$
$\hat{1} = \rho_{+++}$	$>$	$>$	$>$	$>$	$>$	$>$	$>$	$=$

(3.21)

These can be arranged into a lattice, see Fig. (3.2). It may be intuitively obvious that the idempotents are arranged according to the binomial theorem.

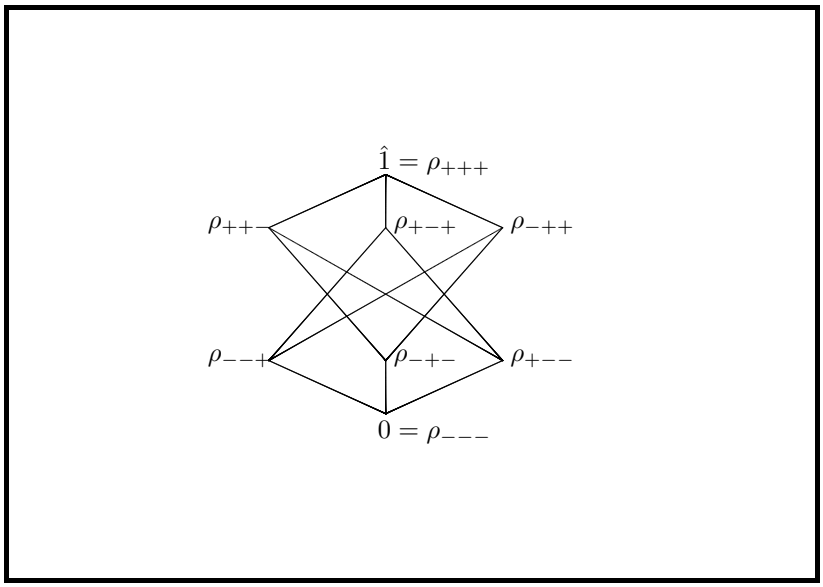


Figure 3.2: Drawing of the lattice of a complete set of commuting idempotents of the algebra of 3×3 matrices, see text, Eq. (3.21). The primitive idempotents are ρ_{--+}, ρ_{-+-}, and ρ_{+--}.

That is, with $N \times N$ matrices there will be $N + 1$ rows, and the number of idempotents in the nth row will be the coefficient of x^n in the expansion of the polynomial $(1 + x)^N$. In the case of Fig. (3.2), $N = 3$ and the four rows of the lattice have $3!/(n!(3 - n)! = 1$, 3, 3, and 1 element each. We will see this pattern in the structure of the elementary particles in that there is one electron, three anti-up quarks, three down quarks, and one neutrino.

We have found the idempotent structure of the diagonal 3×3 matrices; what does this tell us about the structure of more general 3×3 matrices? Let S be any invertible matrix. Then the following transformation:

$$A \to SAS^{-1}. \tag{3.22}$$

preserves multiplication:

$$(SAS^{-1})(SBS^{-1}) = S(AB)S^{-1}, \tag{3.23}$$

and therefore preserves idempotency and roots of unity, and preserves addition:

$$SAS^{-1} + SBS^{-1} = S(A + B)S^{-1}, \tag{3.24}$$

and therefore preserves primitive idempotency and the partial ordering. We will leave it as an exercise for the reader to prove that the relation is general; that is, any set of complete primitive idempotents in 3×3 matrices generates an

idempotency partial ordering that is equivalent to that of the group of diagonal matrices.[1]

Quarks come in three "color" states which we will label as R, G, and B which stand for red, green and blue. Color is completely symmetric under rotation; that is, if we make the replacement:

$$\begin{array}{ccc} R & \rightarrow & G' \\ G & \rightarrow & B' \\ B & \rightarrow & R' \end{array} \tag{3.25}$$

we expect that our physics will remain the same. The above is an even permutation on the colors. We will not require that color symmetry be preserved on odd permutations such as $R \rightarrow R'$, $G \rightarrow B'$, $B \rightarrow G'$.

Suppose there is a process that can take a red particle of type I and convert it to a red particle of type F (which stand for initial and final states), and this process can be modeled with an amplitude of α. By color symmetry, we expect the process to use the same amplitude when the initial state and final states are both green, or when they are both blue. This is needed in order to preserve the above symmetry.

The amplitude α need not be at all similar to the amplitude for the process that takes a red particle in the initial state I and turns it into a green particle in the final state F. Let us suppose that the amplitude β applies to such an interaction. By color symmetry, β must also apply to the process which takes a green particle in the initial state and produces a blue particle. Similarly for a process that takes a blue particle in the initial state and produces a red particle in the final state. Finally, let an amplitude of γ apply to the three remaining conversions. We have defined 9 amplitudes. They can be arranged in a 3×3 table labeled horizontally and vertically by the particle types:

	I_R	I_G	I_B
F_R	α	γ	β
F_G	β	α	γ
F_B	γ	β	α

(3.26)

In the above table, initial states run along the column and final states run along the rows. With this we can easily read off the results when a particle is in a given initial state. For example, if the particle begins in the green state, then the middle column applies, and the process will result in a final red particle with amplitude γ, a green particle with amplitude α, and a blue particle with amplitude β.

Suppose we had a particle that was initially in some superposition of color and we were representing it with a vector with R, G and B components:

$$|I\rangle = \begin{pmatrix} I_R \\ I_G \\ I_B \end{pmatrix}. \tag{3.27}$$

[1] The author figures he could prove this but only inelegantly. If you happen to know how to do it in an elegant fashion, please send me a line at carl@brannenworks.com .

To figure out how this particle is modified by our process, we have to sum the amplitudes for the various possibilities, and add them together. For example, what is the final amplitude for the blue state? There are three contributing processes, $I_R \to F_B$, $I_G \to F_B$, and $I_B \to F_B$. The amplitudes for these three processes are γ, β, and α, respectively. The incoming vector has I_R, I_G, and I_B components, so the total final blue component is:

$$F_B = \gamma I_R + \beta I_G + \alpha I_B. \tag{3.28}$$

Circulant matrices model color processes.

This is just matrix multiplication. Therefore we rewrite our table of amplitudes as a 3×3 matrix:

$$M = \begin{pmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{pmatrix}. \tag{3.29}$$

and write $|F\rangle = M\ |I\rangle$. The matrix in the above equation is a "circulant" 3×3 matrix. Circulant matrices have the convenient properties that sums and products of circulant matrices are also circulant – they form a subalgebra.

States and operators are the same thing.

The philosophy of the state vector formalism is that the fundamental objects in quantum mechanics are states, and these states can be separated, in the above, into initial and final states. This is in contrast to the philosophy of the density operator formalism which treats the density operators as the fundamental objects which share the same algebra as the operators on those objects. This goes both ways. In addition to treating our states as operators, we can also treat our operators as states. This is not at all a trivial extension and we will discuss it in much greater detail in later chapters.

For the moment, let us find the idempotent structure of the circulant 3×3 matrices. Accordingly, suppose that the operator M of Eq. (3.29) is a density operator, and is therefore idempotent:

$$\begin{pmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{pmatrix}^2 = \begin{pmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{pmatrix}. \tag{3.30}$$

This gives three complex equations in three complex unknowns:

$$\begin{aligned} \alpha &= \alpha^2 + 2\beta\gamma, \\ \beta &= \gamma^2 + 2\alpha\beta, \\ \gamma &= \beta^2 + 2\alpha\gamma. \end{aligned} \tag{3.31}$$

Adding all three of the above gives:

$$\alpha + \beta + \gamma = (\alpha + \beta + \gamma)^2, \tag{3.32}$$

and therefore $\alpha+\beta+\gamma = 0$ or $= 1$. For any algebra, if ρ is an idempotent, then so is $1 - \rho$. Translating this into circulant form, this means that if (α, β, γ) gives an idempotent, then so does $(1-\alpha, -\beta, -\gamma)$. Thus we need only consider

values of α, β, and γ that satisfy $\alpha+\beta+\gamma=0$; we can get the other solutions by computing $1-\rho$. Accordingly, we assume that

$$\alpha = -(\beta+\gamma). \tag{3.33}$$

Multiplying the last two equations of Eq. (3.31) by γ and β, respectively, and subtracting gives:

$$\gamma^3 = \beta^3. \tag{3.34}$$

The complex cubed root of one, $e^{2i\pi/3}$, will appear over and over in regard to circulant matrices. Let us have pity on both author and reader and abbreviate this number as ν:

$$\begin{array}{rcccl} \nu & = & e^{+2i\pi/3} & = & -0.5+i\sqrt{0.75}, \\ \nu^* & = & e^{-2i\pi/3} & = & -0.5-i\sqrt{0.75}. \end{array} \tag{3.35}$$

With the ν notation, we can write γ in terms of β as:

$$\gamma = \beta\,\nu^k. \tag{3.36}$$

We can substitute the above equation and Eq. (3.33) into the second equation of Eq. (3.31) to get the following equation for β:

$$\beta = \beta^2\,\nu^{-k} + 2(-\beta-\beta\nu^k)\beta, \tag{3.37}$$

which is solved by either $\beta=0$ or

$$\beta = \nu^k/(1-2\nu^{-k}-2\nu^k). \tag{3.38}$$

If $\beta=0$, then $\gamma=0$ and $\alpha=0$. Thus we have a complete set of solutions to Eq. (3.31) subject to $\alpha+\beta+\gamma=0$. Adding back in the $\alpha+\beta+\gamma=1$ solutions we have the complete solution set as:

α	β	γ
0	0	0
1/3	1/3	1/3
1/3	$\nu/3$	$+\nu^*/3$
1/3	$\nu^*/3$	$+\nu/3$
2/3	$-\nu/3$	$-\nu^*/3$
2/3	$-\nu^*/3$	$-\nu/3$
2/3	$-1/3$	$-1/3$
1	0	0

(3.39)

Circulant 3×3 matrices have the same lattice as 3×3 diagonal matrices.

There are eight solutions. If we write them out in matrix form so that we can perform addition on them, we will see that they define a partial ordering of idempotents that is identical to the one we computed for the diagonal 3×3 matrices,see Fig. (3.2). In fact, the above table is organized in the same order as in Eq. (3.21). The reader is invited to derive a transformation S that diagonalizes the circulant matrices; that is, defines a translation of the form $\chi\rightarrow S\chi S^{-1}$ that takes the diagonal idempotents to the circulant idempotents.

If a set of matrices commute, then they share a set of eigenvectors. This is the case for any set of commuting idempotents, so it certainly applies to both of our 3×3 examples. But circulant matrices in general share common eigenvectors. No matter the values of α, β and γ, the 3×3 circulant matrices have three eigenvectors that can be written as:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \nu \\ \nu^* \end{pmatrix}, \begin{pmatrix} 1 \\ \nu^* \\ \nu \end{pmatrix}. \tag{3.40}$$

Since all circulant matrices share the same eigenvectors, the corresponding eigenvalues can be used to classify the circulant matrices. For the eight idempotents defined in Eq. (3.21), the eigenvalues are as follows:

α	β	γ	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ \nu^* \\ \nu \end{pmatrix}$	$\begin{pmatrix} 1 \\ \nu \\ \nu^* \end{pmatrix}$
0	0	0	0	0	0
1/3	1/3	1/3	1	0	0
1/3	$\nu/3$	$+\nu^*/3$	0	1	0
1/3	$\nu^*/3$	$+\nu/3$	0	0	1
2/3	$-\nu/3$	$-\nu^*/3$	1	0	1
2/3	$-\nu^*/3$	$-\nu/3$	1	1	0
2/3	$-1/3$	$-1/3$	0	1	1
1	0	0	1	1	1

(3.41)

This is another way of writing the structure of the lattice of the circulant matrices.

Chapter 4

Representations

The tale is as old as the Eden Tree—and new as the new-cut tooth—
For each man knows ere his lip-thatch grows he is master of Art and Truth;
And each man hears as the twilight nears, to the beat of his dying heart,
The Devil drum on the darkened pane: "You did it, but was it Art?"

SINCE THIS BOOK IS describing a geometric formulation of quantum mechanics, it is not strictly necessary for us to delve into the subject of representations of Clifford algebras. However, we are very much concerned with the elementary particles, and it turns out that there is a very strong connection between elementary particles and representations.

Accordingly, we will now turn to the task of defining representations from a particle point of view. In doing this, we will find that there are more elegant ways of describing a representation than are typically seen in the literature.

4.1 Clifford Algebras

The number of unit multivectors in a real Clifford algebra with M vectors is 2^M as each vector as each unit multivector can either have zero or one copies of a given vector. Each of these unit multivectors is a degree of freedom for the Clifford algebra, so if the Clifford algebra is real there will be 2^M real degrees of freedom in the algebra, and if the Clifford algebra is complex there will be 2^M complex degrees of freedom or twice that number of real degrees of freedom.

A real $N \times N$ matrix algebra has N^2 real degrees of freedom and a complex $N \times N$ matrix algebra has $2N^2$ real degrees of freedom. Consequently, in making a faithful[1] representation of a Clifford algebra out of matrices, we will choose real or complex matrices according as the number of vectors, M, is even

[1] Faithful here means 1 to 1 and onto.

or odd:

Clifford Algebra	M is even	M is odd
real	$2^{(M/2)}$ real matrices	$2^{(M-1)/2}$ complex
complex	$2^{(M/2)}$ complex	$2^{(M+1)/2}$ real matrices

(4.1)

For example, in the first chapter we treated the Pauli algebra as a real Clifford algebra with 3 vectors, σ_x, σ_y and σ_z; so $M = 3$. This is an odd number of vectors and a real Clifford algebra, so taking the upper right entry in Eq. (4.1), the matrix representation uses complex matrices of size $2^{(3-1)/2} = 2$. Sure enough the Pauli spin matrices are 2×2 complex matrices. The Dirac algebra is usually treated as a complex Clifford algebra with 4 vectors, γ_0, γ_1, γ_2, and γ_3, so $M = 4$. Examining the lower left entry in Eq. (4.1), the matrix representation uses complex matrices of size $2^{(4/2)} = 4$. Sure enough the Dirac gamma matrices are 4×4 complex matrices.

The next least complicated matrix algebra after the 3×3 matricesare the 4×4 matrices. Four is a power of two, and so we can give these matrices a geometric interpretation by considering them as a representation of a Clifford algebra, the Dirac algebra. Physicists are more used to working with representations of the Dirac algebra rather than the algebra itself and the representations are called the "Dirac gamma matrices."

The Dirac gamma matrices are usually designated as γ^0, γ^1, γ^2, and γ^3. These are associated with the four dimensions of space-time, respectively, t, x, y, and z. The numeric designation is convenient for the summation convention. This is particularly convenient when considering the Dirac gradient operator, $\gamma^\mu \partial_\mu$, which is explicitly covariant. This book is mostly concerned with the internal states of particles; for this we will need neither covariance nor that much use of the Dirac operator. On the other hand, we are very concerned with the geometric meaning of quantum mechanics and we will be doing a lot of algebra with these objects.

As a geometric improvement on the usual Dirac notation, we can replace the numbers $0-3$ by letters corresponding to the given direction in the manner of the Pauli algebra. For example instead of γ^3 we could write γ_z. The Dirac algebra would then be distinguished from the Pauli algebra by the use of γ instead of σ. But from the point of view of Clifford algebra, the designation of the vector should be sufficient. That is, it is the vectors that generate and define the Clifford algebra; the Clifford algebra does not define the vectors.

Hat notation: $\hat{x}$, $\hat{y}$, $\hat{z}$, and $\hat{t}$ are an abbreviation for the basis vectors of the Dirac algebra, γ^0, γ^1, γ^2, and γ^3.

Accordingly, we will use a notation that abbreviates the usual gamma matrix notation and covers the Pauli algebra too:

Pauli	Dirac	Clifford
$\times$	γ^0	$\hat{t}$
σ_x	γ^1	$\hat{x}$
σ_y	γ^2	$\hat{y}$
σ_z	γ^3	$\hat{z}$

(4.2)

The reader who takes up this notation will find that it speeds up work considerably. When we need to distinguish between coordinates and Clifford operator

elements, the "hat" will distinguish the Clifford algebra unit vectors. In writing Clifford unit multivectors, we can simply abut the vectors as in $\hat{x}\hat{y}$, or better, we can draw a single hat over the product as in $\widehat{xy}$. If we do draw them with a single hat, we will arrange the elements in a standard order. [2]

In general, an element of an algebra is a sum of scalar multiples of its unit multivectors. For example, any element of the Pauli algebra can be written in the form:

$$\alpha_1\hat{1} + \alpha_x\hat{x} + \alpha_y\hat{y} + ... + \alpha_{xyz}\widehat{xyz}, \tag{4.3}$$

where the α_χ are scalars, that is, real numbers. Because the scalars are taken to be real numbers, the Pauli algebra is a real algebra.

Complex algebras versus real algebras

We can also suppose that the scalars are complex numbers; we then call the algebra a "complex algebra", or an "algebra over the complex numbers". In the first chapter we showed that we could think of the pseudoscalar element of the Pauli algebra, $\widehat{xyz}$, as the imaginary unit of the complex numbers. In doing this we were making a bit of a confusion of the notation because we would be writing $i\hat{1} = i$, and confusing a scalar with the unit operator times that scalar. In order to back away a bit from this confusion, we will extend our hats over the imaginary unit where appropriate. That is, when i is geometric we will write $\hat{i}$ instead of i, and $\widehat{ixy}$ instead of $i\widehat{xy}$.

Many people working in Geometric algebra try to replace all imaginary numbers from their calculations other than the geometric ones. This makes sense from a physical perspective. However, in solving equations in mathematics, imaginary numbers can be very useful, and we can never know for sure that the objects we discuss in physics are mathematical conveniences that might have imaginary numbers present, or physical objects that must be built from real numbers only. For example, imaginary numbers can appear naturally in Fourier transforms. Our notation will allow these sorts of calculations to go forward without more confusion than is inevitable. In such a situation, it is possible to have two imaginary units, both squaring to -1 and both commuting with the entire algebra.

In the Pauli algebra, the element $\widehat{xyz}$ could be thought of as an imaginary unit because it squares to -1 and commutes with all elements in the algebra. Under what conditions can an element commute with the entire algebra?

Let $\widehat{\chi}$ be a unit multivector that we wish to test to see if it commutes with the whole algebra. Since the algebra is made from products of vectors, it is necessary and sufficient that $\widehat{\chi}$ commute with all the vectors. Suppose that $\widehat{\chi}$ is a product of m vectors and the algebra is generated by N vectors.

If $\widehat{\chi}$ is not a scalar then $1 < m$ and we can pick a vector $\hat{v}$ that is part of $\hat{\chi}$. In commuting $\hat{v}$ with $\hat{\chi}$, we will perform m commutations. In $m-1$ of these, the vectors being commuted will be different from $\hat{v}$ and a minus sign will appear because Clifford vectors anticommute. In 1 of these, we will be commuting $\hat{v}$ with itself and no minus sign will appear. Therefore for $\hat{\chi}$ to commute with $\hat{v}$, we must have that $m-1$ is even.

[2] Or send author a nasty note: carl@brannenworks.com .

If $\hat{\chi}$ is not the pseudoscalar, that is, if it is not the product of all the vectors, then $m < N$ and we can pick a vector $\hat{v}$ that is not in the product that makes up $\hat{\chi}$. In this case, our commutation will be performed with m anticommutations and for $\hat{v}$ to commute with $\hat{\chi}$ we must have that m is even. But m and $m-1$ cannot both be even. Therefore any element of a Clifford algebra that commutes with everything in the algebra must be either a scalar or a pseudoscalar.

The scalars always commute with the entire algebra. But the pseudoscalar may or may not. In commuting a vector v with $\widehat{\chi}$, there will be $m = N-1$ anticommutations and one commutation. Therefore the pseudoscalar will commute with the algebra if the algebra has an odd number of basis vectors. And if there is an even number of basis vectors, then the pseudoscalar anticommutes with the basis vectors (and commutes with even products of basis vectors).

To have a geometric imaginary unit, the unit pseudoscalar must, in addition to commuting with the rest of the algebra, also square to -1. This will happen in some Clifford algebras and not in others depending on the signature and the number of basis vectors. First consider the Clifford algebras of positive signature. It can be shown that the unit pseudoscalar in such a Clifford algebra will have positive signature if and only if the number of basis vectors is equivalent to 0 or 1 modulo 4. Example: the Pauli algebra has 3 basis vectors. This is not equivalent to 0 or 1 modulo 4, so the unit pseudoscalar of the Pauli algebra squares to -1: that is, $\widehat{xyz}^2 = -1$.

If a Clifford algebra has an odd number of basis vectors with negative signature, then the signature of the pseudoscalar of that Clifford algebra will be negated with respect to the pseudoscalar of a Clifford algebra of purely positive signature. Example: the Dirac algebra has 4 basis vectors. Four is equivalent to 0 modulo 4 so the unit pseudoscalar of the purely positive Clifford algebra with four vectors will have a positive signature. But the Dirac algebra has an odd number (i.e. one: $\hat{t}$) of basis vectors that have negative signature. So the pseudoscalar of the Dirac algebra instead squares to -1: that is, $\widehat{xyzt}^2 = -1$.

The same rule that tells whether the unit pseudoscalar squares to $+1$ or -1 tells whether any unit multivector of a Clifford algebra squares to ± 1. After a certain amount of use, the reader will find that the rules for determining whether the square of a unit multivector is $+1$ or -1 become intuitive and automatic. Until that time, photocopying Table (4.1) and keeping it handy may assist.

Two Clifford unit multivectors will either commute or anticommute. To find out which, you can anticommute the vectors they are made of but there is an easier way. First of all, in determining whether they commute or anticommute we can start from the assumption that their vectors are ordered in any particular order, so long as we are consistent. Let's begin with the two multivectors a and b written as follows:

$$\begin{aligned} a &= (a_1 a_2 ... a_j)(s_1 s_2 ... s_l), \\ b &= (b_1 b_2 ... b_k)(s_1 s_2 ... s_l), \end{aligned} \tag{4.4}$$

Table 4.1: Signs of squares of Clifford algebra MVs with n_+ positive vectors and n_- negative vectors.

		n_+							
		0	1	2	3	4	5	6	7
	0	+	+	−	−	+	+	−	−
	1	−	+	+	−	−	+	+	−
	2	−	−	+	+	−	−	+	+
	3	+	−	−	+	+	−	−	+
n_-	4	+	+	−	−	+	+	−	−
	5	−	+	+	−	−	+	+	−
	6	−	−	+	+	−	−	+	+
	7	+	−	−	+	+	−	−	+

where the $\{s_n\}$ are l shared vectors, the $\{a_n\}$ are j vectors that only a has, and the $\{b_n\}$ are k vectors that only b has. Then compute:

$$\begin{aligned} ab &= (a_1a_2...a_j)(s_1s_2...s_l)(b_1b_2...b_k)(s_1s_2...s_l), \\ &= (-1)^{kl}(a_1a_2...a_j)(b_1b_2...b_k)(s_1s_2...s_l)(s_1s_2...s_l), \\ &= (-1)^{kl+jk}(b_1b_2...b_k)(a_1a_2...a_j)(s_1s_2...s_l)(s_1s_2...s_l), \\ &= (-1)^{kl+jk+jl}(b_1b_2...b_k)(s_1s_2...s_l)(a_1a_2...a_j)(s_1s_2...s_l), \\ &= (-1)^{kl+jk+jl}\, ba. \end{aligned} \tag{4.5}$$

Multivectors commutation rule.

Therefore the two multivectors commute if $kl+jk+jl$ is even and anticommute if it is odd. So the rule is this: Count the number of vectors that the two unit multivectors share, and the numbers that each has unique. You have three numbers. If more of them are odd then even, then the two unit multivectors anticommute. If more of them are even then odd, then the two unit multivectors commute. Some examples:

A	B	j	k	l	Commute?
$\widehat{xyzt}$	$\widehat{x}$	3	0	1	no
$\widehat{xyz}$	$\widehat{xyt}$	1	1	2	no
$\widehat{xy}$	$\widehat{yz}$	1	1	1	no
$\widehat{xy}$	$\widehat{zt}$	2	2	0	yes
$\widehat{xyz}$	$\hat{x}$	2	0	1	yes
$\widehat{xzt}$	$\hat{y}$	3	1	0	no
$\hat{x}$	$\hat{y}$	1	1	0	no

(4.6)

4.2 The Dirac Algebra

The choice of signature for the basis vectors of a Clifford algebra is not completely arbitrary.[11] For this book we will follow the lead of the Pauli algebra which has positive signature $(+++)$, and give the spatial basis vectors of the Dirac algebra a positive signature. Thus our unit vectors square as follows:

$$\begin{array}{rclcrcl} \hat{t}^2 & = & -1, & \qquad & \hat{x}^2 & = & +1, \\ \hat{y}^2 & = & +1, & \qquad & \hat{z}^2 & = & +1. \end{array} \tag{4.7}$$

This is a less common choice of signature.

As with the Pauli algebra, any product of Dirac algebra vectors can be reduced to $\pm$ a product that includes each vector at most once. This gives 16 Dirac "unit multivectors", twice as many as the Pauli unit multivectors. We will order the Dirac unit multivectors as follows:

Dirac algebra unit multivectors.

$$\begin{array}{cccc} \hat{1}, & \hat{x}, & \hat{y}, & \hat{z}, \\ \hat{t}, & \widehat{xt}, & \widehat{yt}, & \widehat{zt}, \\ \widehat{xyz}, & \widehat{yz}, & \widehat{xz}, & \widehat{xy}, \\ \widehat{xyzt}, & \widehat{yzt}, & \widehat{xzt}, & \widehat{xyt}. \end{array} \tag{4.8}$$

Note that the above has been arranged so that the columns have orientation n, x, y, and z, respectively. The orientation we called "neutral", Eq. (2.13), in the Dirac algebra now has time vectors, but these are present in the other orientations and we will still call it the neutral orientation.

With our choice of signature, 10 of the Dirac unit multivectors square to $+1$ and 6 square to -1:

$$\begin{array}{rl} \{\hat{1}, \hat{x}, \hat{y}, \hat{z}, \widehat{xt}, \widehat{yt}, \widehat{zt}, \widehat{yzt}, \widehat{xzt}, \widehat{xyt}\} & +1, \\ \{\widehat{yz}, \widehat{xz}, \widehat{xy}, \hat{t}, \widehat{xyz}, \widehat{xyzt}\} & \text{-1.} \end{array} \tag{4.9}$$

If we had chosen the opposite signature, we would have only 6 elements square to $+1$ and it would be harder to make roots of unity in a real Dirac algebra.

The Dirac algebra is normally treated as a complex algebra. This means that the Dirac algebra consists of sums of complex scalar multiples of the 16 Dirac unit multivectors. For example,

$$(1+3i)\hat{1} + (2+7i)\widehat{xyz} - 3\widehat{zt} \tag{4.10}$$

is in the complex Dirac algebra. We can also consider the real Dirac algebra, it would consist of sums of real scalar multiples of Dirac unit multivectors. An example of an element in the real Dirac algebra is:

$$2\hat{1} - 5\hat{x} + 3.7\hat{y} + 22\widehat{xyzt} \tag{4.11}$$

For the elementary particles we have need only of the complex Dirac algebra. Our discussion of the real Dirac algebra is only for pedagogical purposes. The real and complex Dirac algebras can be faithfully represented by real and complex 4×4 matrices, respectively.

The Dirac equation is normally written with spinors. In our geometric language, we write it as:

$$\begin{aligned} 0 &= (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{t}\partial_t + m)|\psi\rangle, \\ &= (\nabla + m)\,|\psi\rangle, \end{aligned} \tag{4.12}$$

where we have introduced ∇ as the geometric gradient, the differential part of the Dirac operator. We translate this into density operator form by replacing the ket $|\psi\rangle$ with the product of the corresponding density operator with the vacuum operator: $\rho_\psi\ \rho_0$. The resulting equation is:

$$(\nabla + m)\rho_\psi\ \rho_0. \tag{4.13}$$

In the above equation, as in the first chapter, ρ_0 is a constant primitive idempotent, and is quite arbitrary. Since ρ_0 is constant, it does not interact with the partial derivatives. Since ρ_0 is arbitrary, we can rewrite the above with several different values for ρ_0.

Suppose ρ_0, ρ_0', ρ_0'' and ρ_0''' are a complete set of primitive idempotents of the Dirac algebra so that $\hat{1} = \rho_0 + \rho_0' + \rho_0'' + \rho_0'''$. Then we can write four different Dirac equations (with four different ρ_ψ:

$$\begin{aligned} 0 &= (\nabla + m)\rho_\psi\ \rho_0, \\ 0 &= (\nabla + m)\rho_\psi'\ \rho_0', \\ 0 &= (\nabla + m)\rho_\psi''\ \rho_0'', \\ 0 &= (\nabla + m)\rho_\psi'''\ \rho_0''', \end{aligned} \tag{4.14}$$

Define Ψ as follows:

$$\Psi = \psi\rho_0 + \psi'\rho_0' + \psi''\rho_0'' + \psi'''\rho_0'''. \tag{4.15}$$

Because a complete set of primitive idempotents annihilate one another, we can pick out the ψs from Ψ by right multiplying by the appropriate primitive idempotent:

$$\begin{aligned} \psi &= \Psi\ \rho, \\ \psi' &= \Psi\ \rho', \\ \psi'' &= \Psi\ \rho'', \\ \psi''' &= \Psi\ \rho'''. \end{aligned} \tag{4.16}$$

This means that we can add up the four equations of Eq. (4.14) into a single equation for Ψ:

$$0 = (\nabla + m)\Psi. \tag{4.17}$$

Now the above looks a lot like the usual spinor Dirac equation, but it has four times as many degrees of freedom. It consists of four non interacting Dirac equations. We will call this the "generalized Dirac" equation, and where the context is clear, may leave off the word "generalized". Since the operator version of a wave function has four times as many degrees of freedom, we will use capital letters to designate the operator version as in Ψ, and small letters to designate the spinor version as in ψ.

In particle physics, the Dirac equation is used to compute the propagators of the various spin$-1/2$ fermions. We can think of the four propagators written as above as applying to four different (i.e. "distinguishable") spin$-1/2$ particles. For example, we could assign ψ to the electron (and positron), and then assign ψ', ψ'', and ψ''' to the three colors of the up quark.

If there were only four elementary fermions, we could consider our task of unifying the elementary particles as well under way, but in fact there are instead 24 of them, that is, the electron, the neutrino, three up quarks, three down quarks, and then these eight particles in two more generations. Thus our equation is 6 times too small. While the four spin$-1/2$ particles contained in Ψ cannot be easily assigned to four elementary particles, we will retain this idea and will later apply it to a preon model of the elementary fermions. But the fact that we can assign multiple fermions to a single generalized Dirac equation[3] suggests that we should take a look at geometric transformations of the generalized Dirac equation, and see if these can be related to the internal symmetries of elementary particles.

Suppose that $\iota\chi$ is a unit multivector from the Dirac algebra that happens to square to $\hat{1}$. This could be one of the commuting roots of unity that we could use to define a complete set of primitive idempotents. We can use ι_χ to define a transformation on the Dirac algebra as follows:

$$M \rightarrow \iota_\chi M \iota_\chi. \tag{4.18}$$

The above transformation preserves 0 and $\hat{1}$, and if M and N are in the Dirac algebra, then their product and sum are transformed in a manner that preserves multiplication and addition:

$$\begin{array}{rcl} 0 & \rightarrow & \iota_\chi 0 \iota_\chi = 0, \\ \hat{1} & \rightarrow & \iota_\chi \hat{1} \iota_\chi = \iota_\chi^2 = \hat{1}, \\ M+N & \rightarrow & \iota_\chi (M+N)\iota_\chi = (\iota_\chi M \iota_\chi) + (\iota_\chi N \iota_\chi), \\ MN & \rightarrow & \iota_\chi (MN)\iota_\chi = (\iota_\chi M \iota_\chi)(\iota_\chi M \iota_\chi). \end{array} \tag{4.19}$$

We can also apply this transformation to ∇, and since multiplication and addition are preserved, we can transform a generalized Dirac equation into a new generalized Dirac equation as follows:

$$\begin{array}{rcl} & 0 = & (\nabla + m)\Psi, \\ \rightarrow & 0 = & (\iota_\chi \nabla \iota_\chi + \iota_\chi m \iota_\chi)\iota_\chi \Psi, \end{array} \tag{4.20}$$

where we have multiplied on the right by ι_χ.

Suppose that Ψ is an eigenstate of ι_χ. This could come about, for example, if Ψ were a primitive idempotent (which we associate with the elementary particles) and ι_χ were one of the commuting roots of unity that define that

[3]The problem of defining the Dirac equation in the spinor formalism in geometric form is far more complicated than the density operator formalism shown here. See Baylis[12] for an enlightening discussion of various methods of geometrically defining the Dirac equation for spinors.

primitive idempotent. Then $\iota_\chi\ \Psi_\pm = \pm\Psi_\pm$, with the sign depending on the eigenvalue. This overall sign can be canceled from Eq. (4.20):

$$\begin{aligned} 0 &= (\iota_\chi \nabla \iota_\chi + \iota_\chi m \iota_\chi)\Psi_\pm, \\ &= (\iota_\chi \nabla \iota_\chi \pm m)\Psi_\pm, \\ &= ((\iota_\chi \hat{x} \iota_\chi)\partial_x + (\iota_\chi \hat{y} \iota_\chi)\partial_y + (\iota_\chi \hat{z} \iota_\chi)\partial_z + (\iota_\chi \hat{t} \iota_\chi)\partial_t \pm m)\Psi_\pm. \end{aligned} \tag{4.21}$$

Now $\Psi_\pm$ is a function of (x, y, z, t). And the partial derivatives all have specific values at each position in spacetime that depend on the details of how the wave function depends on position and time (and not on how the wave function is defined in the Clifford algebra). Therefore we can relate symmetries of the wave function to geometric properties of the primitive idempotent $\Psi_\pm$.

As a first example, let's put $\iota_\chi = \widehat{it}$, and find out what sort of properties a wave function might possess, if it is a eigenvector of this operator:

$$\widehat{it}\Psi_{\pm\widehat{it}} = \pm\Psi_{\pm\widehat{it}}, \tag{4.22}$$

and in addition satisfies the Dirac wave equation:

$$0 = (\nabla + m)\Psi_{\pm\widehat{it}}. \tag{4.23}$$

Transform the above equation with $\widehat{it}$ to obtain:

$$\begin{aligned} 0 &= \widehat{it}(\nabla + m)\widehat{it}\widehat{it}\Psi_{\pm\widehat{it}}(x, y, z, t)\widehat{it}, \\ &= (\widehat{it}\hat{x}\widehat{it}\partial_x + \widehat{it}\hat{y}\widehat{it}\partial_y + \widehat{it}\hat{z}\widehat{it}\partial_z + \widehat{it}\hat{t}\widehat{it}\partial_t + \widehat{it}m\widehat{it})\widehat{it}\Psi_{\pm\widehat{it}}(x, y, z, t)\widehat{it}, \\ &= (-\hat{x}\partial_x - \hat{y}\partial_y - \hat{z}\partial_z + \hat{t}\partial_t + m)(\pm\Psi_{\pm\widehat{it}}(x, y, z, t))\widehat{it}. \end{aligned} \tag{4.24}$$

The trailing $\widehat{it}$ can be removed by multiplying on the right by $\widehat{it}$. This removes all the $\widehat{it}$ from the above. To get the differential operator back into ∇ form, make a substitution to the coordinates:

$$x \to -x, \qquad y \to -y, \qquad z \to -z. \tag{4.25}$$

This will negate the partial derivatives with respect to x, y, and z. The result is:

$$0 = (\nabla + m)(\pm\Psi_{\pm\widehat{it}}(-x, -y, -z, t)). \tag{4.26}$$

Comparing with Eq. (4.23), we see that this will happen automatically if

$$\Psi_{\pm\widehat{it}}(x, y, z, t) = \pm\Psi_{\pm\widehat{it}}(-x, -y, -z, t), \tag{4.27}$$

which is just the definition of even (+) and odd (−) parity in a wave function. Therefore, we define the geometric (or "internal") parity operator P as:

Parity as an eigenvalue relation

$$P = \widehat{it}. \tag{4.28}$$

Note that in deriving the above we needed to compute how $\widehat{it}$ transformed the unit vectors only. In the spinor version of this sort of thing one must also figure

out how P changes the spinors, and since the spinors carry arbitrary phases the process is more complicated and less definitive.

There is a subtle distinction between the way we have defined the parity operator P in Eq. (4.28) and how this is done in most of the literature on particle physics. We have defined the parity operator as an operator that lives in the same Clifford algebra as everything else. The more usual practice is to define the parity operator as something which acts on the state, $\psi(x, y, z, t)$ and converts it into a new state $\psi'(x, y, z, t) = \pm\psi(-x, -y, -z, t)$, with a factor of ± 1 included as a consequence of the "internal parity" of the associated particle. In this book, since our wave functions Ψ, contain more than one particle, the parity operator is an operator no different from the rest of the unit multivectors.

We now move on to analyze the transformations of the other two unoriented Dirac unit multivectors, $\widehat{ixyz}$ and $\widehat{ixyzt}$. As before, we assume eigenvectors of these operators:

$$\begin{aligned} \widehat{ixyz}\Psi_{\pm\widehat{ixyz}} &= \pm\Psi_{\pm\widehat{ixyz}}, \\ \widehat{ixyzt}\Psi_{\pm\widehat{ixyzt}} &= \pm\Psi_{\pm\widehat{ixyzt}}. \end{aligned} \tag{4.29}$$

The unoriented Dirac unit multivectors transform the basis vectors as follows:

$\hat{\alpha}$	$\widehat{it}\hat{\alpha}\widehat{it}$	$\widehat{ixyz}\hat{\alpha}\widehat{ixyz}$	$\widehat{ixyzt}\hat{\alpha}\widehat{ixyzt}$
$\hat{x}$	$-\hat{x}$	$+\hat{x}$	$-\hat{x}$
$\hat{y}$	$-\hat{y}$	$+\hat{y}$	$-\hat{y}$
$\hat{z}$	$-\hat{z}$	$+\hat{z}$	$-\hat{z}$
$\hat{t}$	$+\hat{t}$	$-\hat{t}$	$-\hat{t}$

. (4.30)

Applying the same analysis as with $\widehat{it}$, we find that the time reversal operator T is:

$$T = \widehat{ixyz}. \tag{4.31}$$

Time reversal

A wave function $\Psi_{\pm\widehat{ixyz}}$ that is an eigenstate of T with eigenvalue ± 1 and satisfies the Dirac equation, is either negated or not upon the action of time reversal according to the eigenvalue:

$$\begin{aligned} \widehat{ixyz}\Psi_{\pm\widehat{ixyz}} &= \pm\Psi_{\pm\widehat{ixyz}}, \\ \Psi_{\pm\widehat{ixyz}}(x, y, z, t) &= \pm\Psi_{\pm\widehat{ixyz}}(x, y, z, -t). \end{aligned} \tag{4.32}$$

Again, note that our definition of time reversal is different from the usual. And as before, since our states are operators, we need only define time reversal as a transformation of operators, not spinors. For an elementary discussion of the usual time reversal operator when applied to operators as well as fermions and bosons, see [13, sec 4.4].

The remaining unoriented unit multivector, $\widehat{ixyzt}$ anticommutes with all of the vectors and consequently is associated with the combination of parity and time reversal. We stress that this is a state operator that does not commute with our P or T rather than an operation that is performed on wave functions:

PT as an operator on internal states.

$$\widehat{PT} = \widehat{ixyzt}. \tag{4.33}$$

If $\Psi_{\pm\widehat{ixyzt}}$ is an eigenstate of $\widehat{PT}$, and a solution of the generalized Dirac equation, then it satisfies the combined parity and time reversal equation:

$$\Psi_{\pm\widehat{ixyzt}}(x,y,z,t) = \pm\Psi_{\pm\widehat{ixyzt}}(-x,-y,-z,-t). \tag{4.34}$$

This is also the same as the helicity operator we will discuss later.

The other unit multivectors are oriented and correspond to various possible spacetime symmetries of a solution to the Dirac equation. For example, $\widehat{zt}$ anticommutes with $\hat{z}$ and $\hat{t}$ and commutes with the rest of the vectors, and therefore can be associated with wave functions that are even or odd with respect to the combination of time reversal and mirror reflection in the z direction. Some of these symmetries are very important but are best understood in the context of exponential functions of multivectors and we will discuss them later.

4.3 Matrix Representations

The usual way of defining a matrix representation of a Clifford algebra is to specify the matrices corresponding to the basis vectors of the Clifford algebra. One multiplies these together to get the matrices for the various unit multivectors, if desired. This technique doesn't give much of a hint on how one finds the matrix representation to begin with.

Since the density operator formalism associates quantum states with the primitive idempotents, a more natural way of defining a matrix representation is through the primitive idempotents. We will now illustrate the technique by finding a representation for the Dirac algebra in 4×4 matrices.

As with the 3×3 matrices, it is easy to define the diagonal 4×4 primitive idempotent matrices. There are four such matrices. We will call them $\rho_{\pm\pm}$ as follows:

$$\begin{aligned}
\rho_1 = \rho_{--} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho_2 = \rho_{-+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\rho_3 = \rho_{+-} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho_4 = \rho_{++} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{4.35}$$

We will associate the above four primitive idempotent matrices with four primitive idempotent elements of the Dirac algebra. These are the diagonalized states. But this choice is not quite enough to define the representation. We need to choose one further primitive idempotent which we will call ρ_D where "D" stands for "democratic". It is the only idempotent matrix that has all

elements equal:

$$\rho_D = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{4.36}$$

We've specified five $4x4$ primitive idempotent matrices. Together, these matrices define a matrix representation in the following way. Consider products of the form:

$$U_{mn} = 4\rho_m \, \rho_D \, \rho_n, \tag{4.37}$$

where ρ_m and ρ_n are independently any of the four diagonal primitive idempotent matrices. Then the matrix U_{mn} has all its elements zero except for the position (m, n) where it has a one. For example:

Fully specifying a matrix representation requires the diagonalized states plus the democratic state.

$$\begin{aligned} U_{24} &= 4\rho_2 \, \rho_D \, \rho_4, \\ &= 4\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\tfrac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.38}$$

Therefore if we specify these five primitive idempotents, then the representation is defined in that we can give a geometric interpretation of a 4×4 matrix.

Of course we cannot pick any arbitrary set of 5 primitive idempotents and use them to define a representation of the Dirac algebra. The five primitive idempotent Clifford algebra elements we choose have to match the algebra of the corresponding 4×4 primitive idempotent matrix elements of Eq. (4.35) and Eq. (4.36).

For matching the algebras, we do not need to verify that addition matches. The Clifford algebra and the matrix algebra are both linear spaces over their basis elements and addition will be automatically equivalent. It is in the multiplication that there may be differences.

If it were not for the democratic primitive idempotent, ρ_D, our task would be very simple. The diagonal primitive idempotents multiply very simply. Any two different elements annihilate; they multiply to zero. So we need only analyze how the democratic primitive idempotent interacts with the others.

In the Pauli algebra, the diagonal primitive idempotents correspond to the $\vec{z}$ and $-\vec{z}$ directions. The democratic primitive idempotent of the 2×2 matrices corresponds to the Clifford algebra primitive idempotent $0.5(1 + \sigma_x)$. That is:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(1 + \sigma_x). \tag{4.39}$$

There is a clue here. The Pauli diagonal primitive idempotents are perpendicular to the Pauli democratic primitive idempotent. What we need is a generalization of this relation to more complicated Clifford algebras.

In the Pauli algebra, the fact that the two vectors, $\vec{z}$ and $\vec{x}$, are perpendicular, can be written in the geometric density operator language as:

$$P(z \rightarrow x) = 0.5\rho_z = \rho_z \ \rho_x \ \rho_z. \tag{4.40}$$

We need to understand how this comes about so that we can pick out the structural elements of ρ_x that cause it to be allowable as a democratic primitive idempotent when ρ_z and $\bar{\rho_z}$ are chosen as the diagonal primitive idempotents. Accordingly, rewrite the above with ρ_x written out as a sum of unit multivectors:

$$\begin{aligned} P(z \rightarrow x) &= \rho_z \ 0.5(\hat{1} + \hat{x}) \ \rho_z, \\ &= 0.5\rho_z \ \hat{1} \ \rho_z + 0.5\rho_z \ \hat{x} \ \rho_z. \end{aligned} \tag{4.41}$$

In the above form, the transition probability has been written as two expectation values with respect to the quantum state ρ_z. Rewriting these in expectation value form we have:

$$\begin{aligned} P(z \rightarrow x) &= \langle 0.5\hat{1} \rangle_z + \langle 0.5\hat{x} \rangle_z, \\ &= 0.5\rho_z + 0. \end{aligned} \tag{4.42}$$

Thus the transition probability is the expectation value of the scalar part of ρ_x; the vector part of ρ_x has an expectation value of zero.

Now let us return to the 4×4 problem and compute the transition probability for going from a diagonal primitive idempotent to the democratic primitive idempotent. For example:

$$\begin{aligned} P(\rho_2 \rightarrow \rho_D) &= \rho_2 \ \rho_D \ \rho_2, \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ &= \frac{1}{4}\rho_2. \end{aligned} \tag{4.43}$$

Therefore we require that the primitive idempotent from the Clifford algebra that we choose to be represented by ρ_D have an expectation value of $1/4$ with respect to any of the diagonal primitive idempotents of the Clifford algebra. This is different from the value $1/2$ that was obtained in the case of the Pauli algebra, because the primitive idempotents of the Dirac algebra have scalar values of $1/4$ instead of $1/2$.

Putting all this together, we have a procedure for designing a representation of the Dirac algebra. First choose two commuting roots of unity. These define four commuting primitive idempotents. Assign them in any order to the four diagonal primitive idempotents ρ_1, ρ_2, ρ_3, and ρ_4. Second, find the expectation values for the unit multivectors with respect to ρ_1, ρ_2, ρ_3 and ρ_4. Some of these expectation values will be zero. Find a primitive idempotent that can be

written from these zero expectation unit multivectors and assign it to ρ_D. The result will be a representation of the Dirac algebra in 4×4 matrices.

We can make one more improvement to the procedure. The process "find the expectation values for the unit multivectors with respect to ρ_1, ρ_2, ρ_3, and ρ_4" can be time consuming. There is a short cut for this calculation. Write $\rho_{\pm\pm}$ as a product as follows:

$$\rho_{\pm\pm} = 0.25(1 \pm \iota_1)(1 \pm \iota_2), \tag{4.44}$$

where ι_1 and ι_2 are commuting roots of unity. Suppose that ι_1 and ι_2 are Dirac unit multivectors. Let χ be the Dirac unit multivector for which we wish to compute the expectation value. The computation is as follows:

$$\begin{aligned} \langle\chi\rangle_{\pm\pm} &= \rho_{\pm\pm}\, \chi\, \rho_{\pm\pm}, \\ &= 0.25(1 \pm \iota_1)(1 \pm \iota_2)\, \chi\, 0.25(1 \pm \iota_1)(1 \pm \iota_2). \end{aligned} \tag{4.45}$$

To compute the above, let us attempt to bring χ to the front of the product of idempotents, that is, to factor χ out. If χ commutes with ι_1 and ι_2, then we can factor it out and the two primitive idempotents multiply together. But if χ anticommutes with either or both of ι_1 and ι_2, then in factoring out χ we will change the sign of either or both of ι_1 and ι_2. That is, we will change $(1 \pm \iota_1)$ to $(1 \mp \iota_1)$ or the same for ι_2. But there are two copies of $(1 \pm \iota_1)$ in the above, and if one of them is changed to $(1 \mp \iota_1)$ then the two of them will annihilate and the expectation value will be zero.

Therefore, the unit multivectors that we can use to make the democratic primitive idempotent consist exactly of those unit multivectors that anticommute with either of ι_1 or ι_2. Another way of saying the same thing is that we can use any unit multivector that anticommutes with either of ι_1 or ι_2. Since ι_1 and ι_2 form a set of commuting roots of unity, it is clear that we cannot use ι_1, ι_2, or their product, $\iota_1\iota_2$. Beyond that, if there were another (non scalar) unit multivector χ that commuted with both ι_1 and ι_2, we could use either χ or $i\chi$, whichever squares to $+1$, to extend the set of commuting roots of unity from having 2 elements to a set of size 3. Clearly the Dirac algebra can only have two commuting roots of unity, so the simple rule for the choice of democratic primitive idempotent is that we choose a unit multivector that is not included in the set $\{\iota_1, \iota_2, \iota_1\iota_2\}$.

In choosing the democratic primitive idempotent there are a few more subtle points. If we use ι_3 and ι_4 as the commuting roots of unity that define the democratic primitive idempotent, then we have four choices for the primitive idempotent according as we choose the signs in:

$$\rho_D = 0.25(1 \pm \iota_3)(1 \pm \iota_4). \tag{4.46}$$

This means that for any given choice of the commuting roots of unity we have four different possible democratic primitive idempotents. Suppose that we take the positive signs. Then the democratic primitive idempotent becomes:

$$\begin{aligned} \rho_D &= 0.25(1 + \iota_3)(1 + \iota_4), \\ &= 0.25(1 + \iota_3 + \iota_4 + \iota_3\iota_4), \end{aligned} \tag{4.47}$$

so the democratic primitive idempotent actually has three non scalar portions, not two. And we must have that the expectation values of each of these is zero. This means that some pairs of commuting roots of unity for the democratic primitive idempotent, that would seem to be compatible with the pair of commuting roots of unity chosen for the diagonal roots of unity are actually not compatible. We have to check not only the roots of unity, but also their products for compatibility.

A more elegant way of defining whether the sets $\{\iota_1, \iota_2\}$ and $\{\iota_3, \iota_4\}$ are compatible for defining the diagonal and democratic primitive idempotents is to look at the groups generated by these sets under multiplication. We will designate the group generated by multiplication as "$G(\iota_1, \iota_2)$", so the groups we are interested in here have the following elements:

$$\begin{array}{rcl} G(\iota_1, \iota_2) & = & \{\hat{1}, \iota_1, \iota_2, \iota_1\iota_2\}, \\ G(\iota_3, \iota_4) & = & \{\hat{1}, \iota_3, \iota_4, \iota_3\iota_4\}. \end{array} \tag{4.48}$$

Then $\{\iota_1, \iota_2\}$ is compatible with $\{\iota_3, \iota_4\}$ if and only if

$$G(\iota_1, \iota_2) \cap G(\iota_3, \iota_4) = \{\hat{1}\}, \tag{4.49}$$

that is, they are compatible if and only if the only element that their generated groups share is the scalar.

4.4 Dirac's Gamma Matrices

The derivation of this process has been somewhat complicated and probably confusing as well. To solidify the concepts, let us use this technique to work out some representations of the Dirac algebra in 4×4 matrices. The Dirac algebra is generally treated as a complex algebra, however, as an illustration, let us find a real representation of the real Dirac algebra. This restricts us to using only the positive signature unit multivectors listed in Eq. (4.9):

$$\begin{array}{ccccc} \hat{1}, & \hat{x}, & \hat{y}, & \hat{z}, & \widehat{xt}, \\ \widehat{yt}, & \widehat{zt}^2, & \widehat{yzt}, & \widehat{xzt}, & \widehat{xyt}. \end{array} \tag{4.50}$$

Of course we cannot use $\hat{1}$. The remaining 9 unit multivectors commute or anticommute with each other according to the "+" and "−" signs of the following table:

	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\widehat{xt}$	$\widehat{yt}$	$\widehat{zt}$	$\widehat{yzt}$	$\widehat{xzt}$	$\widehat{xyt}$
$\hat{x}$	+	−	−	−	+	+	−	+	+
$\hat{y}$	−	+	−	+	−	+	+	−	+
$\hat{z}$	−	−	+	+	+	−	+	+	−
$\widehat{xt}$	−	+	+	+	−	−	−	+	+
$\widehat{yt}$	+	−	+	−	+	−	+	−	+
$\widehat{zt}$	+	+	−	−	−	+	+	+	−
$\widehat{yzt}$	−	+	+	−	+	+	+	−	−
$\widehat{xzt}$	+	−	+	+	−	+	−	+	−
$\widehat{xyt}$	+	+	−	+	+	−	−	−	+

(4.51)

. If we were making a representation for some particular purpose, we would choose two of these unit multivectors so that our representation would be diagonal for something we were particularly interested in. For example, if we were associating spin$-1/2$ with the $\widehat{iyz}$, $-\widehat{ixz}$, and $\widehat{ixy}$ elements we might want our representation to have $\widehat{zt}$ diagonal, as is traditional. The first two elements that commute in the above table are $\hat{x}$ and $\widehat{yt}$. We will choose these for the commuting roots of unity that define the diagonal primitive idempotents. We can assign the four primitive idempotents to ρ_n in any arbitrary order. There are $4! = 24$ choices. A natural order is:

$$\begin{array}{rcrcl} \rho_{--} & = & \rho_1 & = & 0.25(1-\hat{x})(1-\widehat{yt}), \\ \rho_{-+} & = & \rho_2 & = & 0.25(1-\hat{x})(1+\widehat{yt}), \\ \rho_{+-} & = & \rho_3 & = & 0.25(1+\hat{x})(1-\widehat{yt}), \\ \rho_{++} & = & \rho_4 & = & 0.25(1+\hat{x})(1+\widehat{yt}). \end{array} \tag{4.52}$$

In defining the four diagonal primitive idempotents we've used three unit multivectors; $\hat{x}$, $\widehat{yt}$ and also their product, $\widehat{xyt}$.

For the democratic primitive idempotent we can use any two of the remaining six positive signature unit multivectors:

	$\hat{y}$	$\hat{z}$	$\widehat{xt}$	$\widehat{zt}$	$\widehat{yzt}$	$\widehat{xzt}$
$\hat{y}$	$+$	$-$	$+$	$+$	$+$	$-$
$\hat{z}$	$-$	$+$	$+$	$-$	$+$	$+$
$\widehat{xt}$	$+$	$+$	$+$	$-$	$-$	$+$
$\widehat{zt}$	$+$	$-$	$-$	$+$	$+$	$+$
$\widehat{yzt}$	$+$	$+$	$-$	$+$	$+$	$-$
$\widehat{xzt}$	$+$	$+$	$+$	$+$	$-$	$+$

. (4.53)

There are nine pairs of commuting roots of unity:

$$\begin{array}{ccccc} \{\hat{y},\widehat{xt}\}, & \{\hat{y},\widehat{zt}\}, & \{\hat{y},\widehat{yzt}\}, & \{\hat{z},\widehat{xt}\}, & \{\hat{z},\widehat{yzt}\}, \\ \{\hat{z},\widehat{xzt}\}, & \{\widehat{xt},\widehat{xzt}\}, & \{\widehat{zt},\widehat{yzt}\}, & \{\widehat{zt},\widehat{xzt}\}. & \end{array} \tag{4.54}$$

Some of these can't be used because the groups they generate include non scalar elements that are in $G(\iota_1,\iota_2) = G(\hat{x},\widehat{yt}) = \{\hat{1},\hat{x},\widehat{yt},\widehat{xyt}\}$. Since $\hat{y}\ \widehat{xt} = -\widehat{xyt}$ and $\widehat{xyt} \in G(\hat{x},\widehat{yt})$, we cannot use the pair $\{\hat{y},\widehat{xt}\}$. Since $\hat{z}\ \widehat{yzt} = -\widehat{yt}$, and $\widehat{yt} \in G(\hat{x},\widehat{yt})$, we cannot use the pair $\{\hat{z},\widehat{yzt}\}$. Finally, $\widehat{zt}\ \widehat{xzt} = \hat{x}$, so we cannot use the pair $\{\widehat{zt},\widehat{xzt}\}$. This leaves six viable pairs for creating the democratic primitive idempotent:

$$\begin{array}{ccc} \{\hat{y},\widehat{zt}\}, & \{\hat{y},\widehat{yzt}\}, & \{\hat{z},\widehat{xt}\}, \\ \{\hat{z},\widehat{xzt}\}, & \{\hat{xt},\widehat{xzt}\}, & \{\hat{zt},\widehat{yzt}\}. \end{array} \tag{4.55}$$

We will choose the first of these pairs, $\{\hat{y},\widehat{zt}\}$. We now have to choose which of the four possible primitive idempotents to use, $\rho_D = 0.25(1\pm\hat{y})(1\pm\widehat{zt})$. We

will take the ++ case. This defines our representation of the Dirac algebra:

$$\begin{aligned}
\rho_1 &= 0.25(1-\hat{x})(1-\widehat{yt}),\\
\rho_2 &= 0.25(1-\hat{x})(1+\widehat{yt}),\\
\rho_3 &= 0.25(1+\hat{x})(1-\widehat{yt}),\\
\rho_4 &= 0.25(1+\hat{x})(1+\widehat{yt}).\\
\rho_D &= 0.25(1+\hat{y})(1+\widehat{zt}).
\end{aligned} \tag{4.56}$$

After deciding that we would have $\hat{x}$ and $\widehat{yt}$ diagonal, we had $4! = 24$ choices for the order of ρ_1, ρ_2, ρ_3 and ρ_4. Then we had six pairs of commuting roots of unity available for the democratic primitive idempotent. Finally we had four sign choices for the democratic primitive idempotent. There are therefore $24 \times 6 \times 4 = 24^2 = 576$ representations of the Dirac algebra that diagonalize $\hat{x}$ and $\widehat{yt}$ (and use only unit multivectors).

Our representation, Eq. (4.56), is written in the geometric language and is quite elegant. To convert it into matrix form, one could use a fairly large amount of algebra but there is a simple trick. First, since we have defined the diagonal primitive idempotents, we can quickly find the solution for this part of the matrix. To simplify our notation for the off diagonal elements, let U_{nm} designate the matrix which is all zero except for a 1 in position (n, m) and rewrite Eq. (4.56) as:

$$\begin{aligned}
U_{11} &= 0.25(\hat{1}-\hat{x}-\widehat{yt}+\widehat{xyt}),\\
U_{22} &= 0.25(\hat{1}-\hat{x}+\widehat{yt}-\widehat{xyt}),\\
U_{33} &= 0.25(\hat{1}+\hat{x}-\widehat{yt}-\widehat{xyt}),\\
U_{44} &= 0.25(\hat{1}+\hat{x}+\widehat{yt}+\widehat{xyt}).
\end{aligned} \tag{4.57}$$

The above amounts to four equations in four unknowns. But they are easier to solve then one might expect:

$$\begin{aligned}
+U_{11}+U_{22}+U_{33}+U_{44} &= \hat{1},\\
-U_{11}-U_{22}+U_{33}+U_{44} &= \hat{x},\\
-U_{11}+U_{22}-U_{33}+U_{44} &= \widehat{yt},\\
+U_{11}-U_{22}-U_{33}+U_{44} &= \widehat{xyt},
\end{aligned} \tag{4.58}$$

The same principle will be applied to the off diagonal elements.

The off diagonal terms will have a similar form. To find them, first note that we can write

$$\begin{aligned}
M_{mn} &= 4\rho_m\,\rho_D\,\rho_n,\\
&= \rho_m(\hat{1}+\hat{y}+\widehat{zt}+\widehat{yzt})\rho_n,\\
&= \rho_m\,\rho_n+\rho_m(\hat{y}+\widehat{zt}+\widehat{yzt})\rho_n,\\
&= 0+\rho_m(\hat{y}+\widehat{zt}+\widehat{yzt})\rho_n.
\end{aligned} \tag{4.59}$$

The zero in the above follows from the fact that we are considering off diagonal terms so $m \neq n$ and so ρ_m will annihilate ρ_n. We will factor the three inner unit multivectors, $\hat{y}$, $\widehat{zt}$, and $\widehat{yzt}$ to the left side of ρ_m. Since these terms do

not commute with ρ_m, doing this will change ρ_m. The change will be that the signs of the commuting roots of unity that make up ρ_m will be changed if that commuting root of unity does not commute with the unit multivector we are factoring out. The result of these sign changes will be that ρ_m will be converted to some other ρ'_m. Sometimes ρ'_m will be equal to ρ_n and these terms will not annihilate.

This is easier done than described. First we make a table showing which of the unit multivectors we will factor, commute with which of the roots of unity of the diagonal primitive idempotents:

	$\hat{x}$	$\widehat{yt}$
$\hat{y}$	$-$	$-$
$\widehat{zt}$	$+$	$-$
$\widehat{yzt}$	$-$	$+$

(4.60)

With the above, we can read off how ρ_m will be altered by the factoring out of our three terms:

	$\hat{y}$	$\widehat{zt}$	$\widehat{yzt}$
$\rho_{--} \rightarrow$	ρ_{++}	ρ_{-+}	ρ_{+-}
$\rho_{-+} \rightarrow$	ρ_{+-}	ρ_{--}	ρ_{++}
$\rho_{+-} \rightarrow$	ρ_{-+}	ρ_{++}	ρ_{--}
$\rho_{++} \rightarrow$	ρ_{--}	ρ_{+-}	ρ_{-+}

. (4.61)

The above was obtained by negating signs in the ρ_m on the left according to the table of Eq. (4.60). It's easier to use if we rewrite it as:

	$\hat{y}$	$\widehat{zt}$	$\widehat{yzt}$
$\rho_1 \rightarrow$	ρ_4	ρ_2	ρ_3
$\rho_2 \rightarrow$	ρ_3	ρ_1	ρ_4
$\rho_3 \rightarrow$	ρ_2	ρ_4	ρ_1
$\rho_4 \rightarrow$	ρ_1	ρ_3	ρ_2

. (4.62)

We can read off the terms that are not annihilated in the off diagonal elements from the above. The action of $\hat{y}$ is to convert $\rho_1 \rightarrow \rho_4$, $\rho_2 \rightarrow \rho_3$, $\rho_3 \rightarrow \rho_2$ and $\rho_4 \rightarrow \rho_1$. Therefore the matrix elements that $\hat{y}$ will survive in are U_{14}, U_{23}, U_{32}, and U_{41}:

$$\begin{aligned}
U_{14} &= \hat{y}\rho_4 = 0.25(+\hat{y} - \widehat{xy} + \hat{t} - \widehat{xt}), \\
U_{23} &= \hat{y}\rho_3 = 0.25(+\hat{y} - \widehat{xy} - \hat{t} + \widehat{xt}), \\
U_{32} &= \hat{y}\rho_2 = 0.25(+\hat{y} + \widehat{xy} + \hat{t} + \widehat{xt}), \\
U_{41} &= \hat{y}\rho_1 = 0.25(+\hat{y} + \widehat{xy} - \hat{t} - \widehat{xt}).
\end{aligned} \tag{4.63}$$

As with the diagonal terms, the above are four equations in four unknowns that have an easy to find solution:

$$\begin{aligned}
+U_{14} + U_{23} + U_{32} + U_{41} &= \hat{y}, \\
-U_{14} - U_{23} + U_{32} + U_{41} &= \widehat{xy}, \\
+U_{14} - U_{23} + U_{32} - U_{41} &= \hat{t}, \\
-U_{14} + U_{23} + U_{32} - U_{41} &= \widehat{xt}.
\end{aligned} \tag{4.64}$$

At this point in time, we've found the matrix representation for $\hat{x}$ in Eq. (4.58), and the representations for $\hat{y}$ and $\hat{t}$ in Eq. (4.64). All that is left is $\hat{z}$, which we can find by treating $\widehat{yzt}$ in the same manner as we just did $\hat{y}$:

$$\begin{array}{rcccl} U_{13} & = & \widehat{yzt}\rho_4 & = & 0.25(+\widehat{yzt} - \widehat{xyzt} + \hat{z} - \widehat{xz}), \\ U_{24} & = & \widehat{yzt}\rho_3 & = & 0.25(+\widehat{yzt} - \widehat{xyzt} - \hat{z} + \widehat{xz}), \\ U_{31} & = & \widehat{yzt}\rho_2 & = & 0.25(+\widehat{yzt} + \widehat{xyzt} + \hat{z} + \widehat{xz}), \\ U_{42} & = & \widehat{yzt}\rho_1 & = & 0.25(+\widehat{yzt} + \widehat{xyzt} - \hat{z} - \widehat{xz}). \end{array} \tag{4.65}$$

again four equations in four unknowns with the simple solution:

$$\begin{array}{rcl} +U_{13} + U_{24} + U_{31} + U_{42} & = & \widehat{yzt}, \\ -U_{13} - U_{24} + U_{31} + U_{42} & = & \widehat{xyzt}, \\ +U_{13} - U_{24} + U_{31} - U_{42} & = & \hat{z}, \\ -U_{13} + U_{24} + U_{31} - U_{42} & = & \widehat{xz}. \end{array} \tag{4.66}$$

We now have matrix representations for all four basis vectors. Gathering them together:

$$\begin{array}{rcl} \hat{x} & = & -U_{11} - U_{22} + U_{33} + U_{44}, \\ \hat{y} & = & +U_{14} + U_{23} + U_{32} + U_{41}, \\ \hat{z} & = & +U_{13} - U_{24} + U_{31} - U_{42}, \\ \hat{t} & = & +U_{14} - U_{23} + U_{32} - U_{41}. \end{array} \tag{4.67}$$

In the usual matrix notation:

$$\begin{array}{c} \hat{x} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{z} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{t} = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \end{array} \tag{4.68}$$

As claimed, the above is a real representation of the Dirac algebra.

4.5 Traditional Gamma Representations

Given a representation of Dirac's gamma matrices, we can extract the geometric description of it by reversing the above procedure. We need only find which unit multivectors are diagonal, and find the democratic primitive idempotent.

A common definition of the gamma matrices[14] is:

$$\begin{array}{cc} \hat{x} = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} & \hat{y} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \\ \hat{z} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} & \hat{t} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \tag{4.69}$$

Note that the above has signature the opposite of the one that we are using. One of the diagonal roots of unity is given above, $\hat{t}$.

To get the other diagonal root of unity, note that $\hat{x}$ and $\hat{y}$ multiply together to give a diagonal matrix. But $\widehat{xy}$ squares to -1, so the other diagonal root of unity is:

$$\widehat{ixy} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.70}$$

There is only one minor problem. The signs are reversed. To put these into our (somewhat arbitrary) standard form that has ρ_{--} the diagonal primitive idempotent with 1 in the top left corner, we need to negate $\hat{t}$ and $\widehat{ixy}$. Later we will associate this root with spin, and the root $\hat{t}$ with parity. Therefore the commuting roots of unity are:

$$\begin{array}{rcl} \iota_1 & = & -\hat{t}, \\ \iota_2 & = & -\widehat{ixy}. \end{array} \tag{4.71}$$

Thus the diagonal primitive idempotents are:

$$\rho_{\pm\pm} = 0.25(1 \pm -\hat{t})(1 \pm -\widehat{ixy}). \tag{4.72}$$

The democratic primitive idempotent takes a bit more work.

Computing $\widehat{xt} = \hat{x}\hat{t}$ gives:

$$\begin{array}{rcl} \widehat{xt} & = & \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ so} \\ -\widehat{xt} & = & \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}. \end{array} \tag{4.73}$$

The same sort of inspiration suggests computing $\widehat{iyz}$ to obtain:

$$\widehat{iyz} = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix} \tag{4.74}$$

These give 2 out of the 3 off diagonal sets of elements that we need. These 3 are going to appear in the geometric definition of the diagonal primitive idempotent. The fourth term that appears in the diagonal primitive idempotent is just the scalar. Therefore, we know that these two unit multivectors, $-\widehat{xt}$ and $\widehat{iyz}$ are two of the commuting roots of unity. In fact, they do commute and square to $+1$. Therefore, the third term we need is simply their product:

$$-\widehat{ixyzt} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}. \tag{4.75}$$

Note that the above is commonly written as $\gamma^5 = i\hat{t}\hat{x}\hat{y}\hat{z}$, the minus sign comes about because we prefer to commute $\hat{t}$ around to the right. The result is that we have:

$$\rho_D = 0.25(\hat{1} - \widehat{xt} + \widehat{iyz} - \widehat{ixyzt}). \tag{4.76}$$

There are three commuting roots of unity associated with the democratic primitive idempotent, $\widehat{xt}$, $\widehat{iyz}$ and $\widehat{ixyzt}$. Since we will associate $\widehat{ixy}$ with spin in the $z-$ direction, we will associate $\widehat{iyz}$ with spin in the $x-$direction. Note that spin in the z and x directions are incompatible measurements, as is always the case between a pair of commuting roots of unity when one is chosen from the democratic primitive idempotent and the other from the diagonal primitive idempotent. The operator $\widehat{ixyzt}$ will be associated with "helicity", and the operator $\widehat{xt}$ will be associated with velocity (note, velocity, *not* momentum) in the x direction.

The complete geometric description of the representation is:

$$\begin{array}{rcrcl} \rho_1 & = & \rho_{--} & = & 0.25(\hat{1} - -\hat{t})(\hat{1} - -\widehat{ixy}), \\ \rho_2 & = & \rho_{-+} & = & 0.25(\hat{1} - -\hat{t})(\hat{1} + -\widehat{ixy}), \\ \rho_3 & = & \rho_{+-} & = & 0.25(\hat{1} + -\hat{t})(\hat{1} - -\widehat{ixy}), \\ \rho_4 & = & \rho_{++} & = & 0.25(\hat{1} + -\hat{t})(\hat{1} + -\widehat{ixy}), \\ \rho_D & = & & & 0.25(\hat{1} - \widehat{xt})(\hat{1} + \widehat{iyz}). \end{array} \tag{4.77}$$

The group formed by the commuting roots of unity for the diagonal primitive idempotents is $G(-\hat{t}, -\widehat{ixy}) = \{\hat{1}, -\hat{t}, -\widehat{ixy}, \widehat{ixyt}\}$. Note that, as we required in our analysis of matrix representations of Clifford algebras, this group has no non scalar elements in common with the group formed by the commuting roots of unity for the democratic primitive idempotent, $G(-\widehat{xt}, \widehat{iyz})$. It is interesting to note that to convert the above representation into a representation for the

signature $-+++$ we need only modify one of the four commuting roots of unity, that is, we replaced $-\hat{t}$ with $\widehat{it}$ or $-\widehat{it}$.

A slightly less common representation of the Dirac algebra is the "chiral representation":

$$\begin{aligned}
\hat{x} &= \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \hat{y} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \\
\hat{z} &= \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} \quad \hat{t} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{4.78}$$

The above representation receives its name because the helicity operator, $\gamma^5 = i\hat{x}\hat{y}\hat{z} = -\widehat{ixyzt}$ is diagonalized:[4]

$$\gamma_5 = -\widehat{ixyzt} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{4.79}$$

The $\hat{x}$, $\hat{y}$, and $\hat{z}$ representatives are unchanged so the other diagonal operator is $\widehat{ixy}$. As before, the signs are reversed and to put them into our standard geometric form requires the roots of unity as follows:

$$\begin{aligned}
\iota_1 &= -\widehat{ixyzt}, \\
\iota_2 &= -\widehat{ixy}.
\end{aligned} \tag{4.80}$$

Thus the diagonal primitive idempotents are:

$$\rho_{\pm\pm} = 0.25(1 \pm -\widehat{ixyzt})(1 \pm -\widehat{ixy}). \tag{4.81}$$

We continue on to the democratic primitive idempotent.

As before, we look for the unit multivectors that are off diagonal and constant. An easy catch is $\hat{t}$. And as before, $\widehat{iyz}$ is such:

$$\widehat{iyz} = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix}. \tag{4.82}$$

The are commuting roots of unity, and as before, so is there product:

$$\widehat{iyzt} = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.83}$$

[4]One would suppose it should be called the "helicity representation".

Thus we have the democratic primitive idempotent of this representation as:

$$\rho_D = 0.25(\hat{1} + \hat{t} + \widehat{iyz} + \widehat{iyzt}). \tag{4.84}$$

and the complete geometric description of the chiral representation is:

$$\begin{array}{rcrcl}
\rho_1 & = & \rho_{--} & = & 0.25(\hat{1} - -\widehat{ixyzt})(\hat{1} - -\widehat{ixy}), \\
\rho_2 & = & \rho_{-+} & = & 0.25(\hat{1} - -\widehat{ixyzt})(\hat{1} + -\widehat{ixy}), \\
\rho_3 & = & \rho_{+-} & = & 0.25(\hat{1} + -\widehat{ixyzt})(\hat{1} - -\widehat{ixy}), \\
\rho_4 & = & \rho_{++} & = & 0.25(\hat{1} + -\widehat{ixyzt})(\hat{1} + -\widehat{ixy}), \\
\rho_D & = & & & 0.25(\hat{1} + \hat{t})(\hat{1} + \widehat{iyz}).
\end{array} \tag{4.85}$$

As usual, the groups formed by the commuting roots of unity for the diagonal and democratic primitive idempotents share only the scalar. Note that in comparison with the usual representation, the above swaps the parity $\widehat{it}$, and helicity $\widehat{ixyzt}$ unit multivectors. We now return to the usual signature, $-+++$.

Chapter 5

Algebra Tricks

His house is Gee & Tellus' Sons,—so goes his jest with men—
He sold us Zeus knows what last year; he'll take us in again.
Disguised behind a livery-team, fur-coated, rubber-shod—
Yet Apis from the bull-pen lows—he knows his brother God!

Now down the lines of tasselled pines the yearning whispers wake—
Pithys of old thy love behold. Come in for Hermes' sake!
How long since that so-Boston boot with reeling Mænads ran?
Numen adest! Let be the rest. Pipe and we pay, O Pan.

IN ORDER TO MAINTAIN a tight connection to the underlying geometry, most of our calculations will be accomplished in the geometric language. Since quantum theory is usually taught by choosing representations of symmetry groups or algebras, these geometric techniques will be unfamiliar to many readers. For convenience, we've collected together some of the more common ones in this chapter.

5.1 Exponentials and Transformations

Define the exponential of a Clifford algebraic element M as usual:

$$\exp(M) = \hat{1} + M + M^2/2! + M^3/3! + ... \tag{5.1}$$

Since the exponent of a matrix is defined, and since one can represent a Clifford algebra in matrix form, the above is well defined. We can check to see what happens when M is various things we've been working on.

Let ι be a root of unity, let a be a real number and compute $\exp(a\iota)$:

$$\begin{aligned} e^{a\iota} &= \hat{1} + a\iota + a^2\iota^2/2! + a^3\iota^3/3! + ..., \\ &= \hat{1} + a\iota + a^2\hat{1}/2! + a^3\iota/3! + ..., \\ &= (1 + a^2/2! + ...)\hat{1} + (a + a^3/3! + ...)\iota, \\ &= \cosh(a)\hat{1} + \sinh(a)\iota. \end{aligned} \tag{5.2}$$

Similarly, if χ_- squares to -1 and a is a real number,

$$e^{a\chi_-} = \cos(a)\hat{1} + \sin(a)\chi_-. \tag{5.3}$$

If we limit ourselves to the subalgebra generated by χ_-, that is, elements of the form

$$a_r\hat{1} + a_i\chi_-, \tag{5.4}$$

where a_r and a_i are real numbers, we can treat χ_- as an imaginary unit. For example, given two numbers of the above sort, their product:

$$(a_r\hat{1} + a_i\chi_-)(b_r\hat{1} + b_i\chi_-) = (a_rb_r - a_ib_i)\hat{1} + (a_ib_r + a_rb_i)\chi_-, \tag{5.5}$$

acts just like the product of two complex numbers:

$$(a_r + ia_i)(b_r + ib_i) = (a_rb_r - a_ib_i) + i(a_rb_i + a_ib_r). \tag{5.6}$$

This means that we can bring in the machinery of complex numbers in making calculations.[1]

In defining complete sets of primitive idempotents we've been using complete sets of commuting roots of unity. Our roots of unity have been quite pedestrian, $\hat{x}$, $\widehat{yt}$, etc. The Dirac algebra includes much more general roots of unity. For example, the following elements all square to $+1$:

$$\begin{array}{ll} 0.6\hat{x} + 0.8\hat{y}, & 2.6\hat{z} - 2.4\hat{t}, \\ 0.6\widehat{yt} + 0.8\hat{y}, & 2.6\hat{x} - 2.4\widehat{yzt}. \end{array} \tag{5.7}$$

The elements in the right column are of "mixed signature", that is, they are a sum over unit multivectors of different signature. The elements in the top row are made from vectors, but the elements in the bottom row are multivectors. In coming up with example multivector roots of unity of the sort above, one must choose two unit multivectors that imitate two unit vectors, in that they must anticommute.

Among these more general roots of unity, one can find sets of commuting roots, and from these one can define complete sets of primitive idempotents. Such complete sets can then be used to produce interesting representations of the Dirac algebra that you can use to pester your students or annoy the readers of your papers. It is a mathematical theorem of Clifford algebra that any complete set of primitive idempotents for a given Clifford algebra has the same lattice, so this would seem to be an unnecessary complication. However, we are associating the basis vectors with coordinates for spacetime and this gives a physical interpretation for such transformations.

If we replace $\hat{x}$ with $0.6\hat{x} + 0.8\hat{y}$ (and modify $\hat{y}$ accordingly), we recognize this transformation as equivalent to a rotation around the $\hat{z}$ axis. That is,

[1] This book is mostly concerned with discrete symmetries of particles so we do not get into complex analysis (i.e. things involving integration and differentiation) here. However, it should be mentioned that David Hestenes' works[15] on the "geometric calculus" does this in a way far more elegant, powerful, and geometrically natural than one would suppose from the simple method shown here.

modifying the coordinates by rotating the unit vectors around the z axis will have this effect on the vectors. This is the obvious physical interpretation of the upper left root of unity shown in Eq. (5.7). Similarly, if we perform a boost in the $-z$ direction to a coordinate system moving with respect to the original one, we expect to see the z and t coordinates mixed in the manner seen in the upper right root of unity of Eq. (5.7).

We can produce transformations of the roots of unity (and therefore the primitive idempotents, and so the quantum states) by using exponential functions as we will get to later in this section. For the top row of Eq. (5.7) these will physically correspond to rotations and boosts. However, in doing this we will also obtain transformations that move the usual roots of unity to the mixed objects of the bottom row and these are more difficult to give a physical interpretation. Nevertheless, the same mathematics that produces the boosts and rotations also allow these, and we must consider the possibility that they have a physical interpretation.

As a first example of an exponential transformation, consider the unit multivector $\widehat{xy}$, or rather, a real multiple of it, $a\widehat{xy}$. Since $\widehat{xy}$ squares to -1, the exponential is:

$$e^{\pm a\widehat{xy}} = \cos(a)\hat{1} \pm \sin(a)\widehat{xy}. \tag{5.8}$$

Since $\widehat{xy}$ commutes with $\hat{z}$ and $\hat{t}$, so does $\exp(a\widehat{xy})$. But it anticommutes with $\hat{x}$ and $\hat{y}$. Consider transformations on a Clifford algebra's elements of the following form:

$$M \rightarrow e^{+a\widehat{xy}}\, M\, e^{-a\widehat{xy}}, \tag{5.9}$$

where M is an element of the Clifford algebra. Note that the above transformation is compatible with addition and multiplication on the Clifford algebra. That is,

$$\begin{aligned} M+N &\rightarrow e^{-a\widehat{xy}}\,(M+N)\,e^{+a\widehat{xy}} &&= e^{-a\widehat{xy}}\,M\,e^{+a\widehat{xy}} + e^{-a\widehat{xy}}\,N\,e^{+a\widehat{xy}}, \\ MN &\rightarrow e^{-a\widehat{xy}}\,(MN)\,e^{+a\widehat{xy}} &&= e^{-a\widehat{xy}}\,M\,e^{+a\widehat{xy}}\,e^{-a\widehat{xy}}\,N\,e^{+a\widehat{xy}}, \end{aligned} \tag{5.10}$$

and that the transformation takes 0 to 0 and $\hat{1}$ to $\hat{1}$.

The notions of primitive idempotents, commuting roots of unity, (and everything else we discuss in the first three chapters of this book) have been defined only through addition and multiplication, so the transformation of Eq. (5.9) will map complete sets of primitive idempotents to complete sets of primitive idempotents, and it will map complete sets of commuting roots of unity to complete sets of commuting roots of unity.

Any element of the Clifford algebra can be described in terms of multiplication and addition of basis vectors, so, since the transformation is compatible with multiplication and addition, we can fully understand its effect on a Clifford algebra by seeing what it does to the basis vectors. As usual, we compute the following by again using the method of factoring the basis vectors out to the left, but now, instead of having primitive idempotents annihilate against each other, we have two exponentials that either cancel or multiply. The $\hat{x}$

vector is transformed by rotating it towards $\hat{y}$:

$$\begin{aligned} \hat{x} \quad \rightarrow \quad & (\cos(a)\hat{1} - \sin(a)\widehat{xy})\hat{x}(\cos(a)\hat{1} + \sin(a)\widehat{xy}), \\ & = \hat{x}(\cos(a)\hat{1} + \sin(a)\widehat{xy})(\cos(a)\hat{1} + \sin(a)\widehat{xy}), \\ & = \hat{x}e^{2a\widehat{xy}}, \\ & = \hat{x}(\cos(2a)\hat{1} + \sin(2a)\widehat{xy}), \\ & = \cos(2a)\hat{x} + \sin(2a)\hat{y}. \end{aligned} \tag{5.11}$$

Similarly, the $\hat{y}$ vector is transformed by rotating it towards $-\hat{x}$:

$$\begin{aligned} \hat{y} \quad \rightarrow \quad & (\cos(a)\hat{1} - \sin(a)\widehat{xy})\hat{y}(\cos(a)\hat{1} + \sin(a)\widehat{xy}), \\ & = \hat{y}(\cos(2a)\hat{1} + \sin(2a)\widehat{xy}), \\ & = \cos(2a)\hat{y} - \sin(2a)\hat{x}. \end{aligned} \tag{5.12}$$

Since $\hat{z}$ and $\hat{t}$ commute with $\widehat{xy}$, they are not effected by the transformation. Thus we see that this transformation is a rotation around an axis through the $\hat{z}$ direction that sends the $\hat{x}$ vector towards the $\hat{y}$ vector.

Similarly, the unit multivector $\widehat{zt}$ generates a boost in the $\hat{z}$ direction. The exponential of $a\widehat{zt}$ gives a hyperbolic function:

$$e^{a\widehat{zt}} = \cosh(a)\hat{1} + \sinh(a)\widehat{zt}. \tag{5.13}$$

This commutes with $\hat{x}$ and $\hat{y}$ so the associated transformation:

$$M \rightarrow e^{-a\widehat{zt}}\, M\, e^{+a\widehat{zt}}, \tag{5.14}$$

leaves $\hat{x}$ and $\hat{y}$ unchanged. To find the effect on $\hat{z}$ and $\hat{t}$, let us introduce a new way of factoring that works with any analytic function of a Clifford algebra element, and which will speed up this sort of calculation:

$$\begin{aligned} f(+\widehat{zt})\,\hat{x} &= \hat{x}\, f(+\widehat{zt}), \\ f(+\widehat{zt})\,\hat{y} &= \hat{y}\, f(+\widehat{zt}), \\ f(+\widehat{zt})\,\hat{z} &= \hat{z}\, f(-\widehat{zt}), \\ f(+\widehat{zt})\,\hat{t} &= \hat{t}\, f(-\widehat{zt}). \end{aligned} \tag{5.15}$$

That is, so long as you can write the function f as a series, the odd terms will anticommute and the even terms will commute. This will give the factorization shown above. Thus the effect of the transformation on $\hat{z}$:

$$\begin{aligned} \hat{z} \quad \rightarrow \quad & e^{-a\widehat{zt}}\, \hat{z}\, e^{+a\widehat{zt}}, \\ & = \hat{z}\, e^{+2a\widehat{zt}} = \hat{z}(\cosh(2a) + \sinh(2a)\widehat{zt}), \\ & = \cosh(2a)\hat{z} + \sinh(2a)\hat{t}. \end{aligned} \tag{5.16}$$

Similarly, the transformation on $\hat{t}$ is:

$$\begin{aligned} \hat{t} \quad \rightarrow \quad & \hat{t}\, e^{+2a\widehat{zt}} = \hat{t}(\cosh(2a) + \sinh(2a)\widehat{zt}), \\ & = \cosh(2a)\hat{z} + \sinh(2a)\hat{t}. \end{aligned} \tag{5.17}$$

Let v be a velocity between 0 and c. Define a by

$$a = 0.5 \tanh^{-1}(v/c). \tag{5.18}$$

Then $\tanh(2a) = v/c$. From trigonometry, we have:

$$\begin{array}{rcl} \cosh(2a) & = & 1/\sqrt{(}1 - v^2/c^2), \\ \sinh(2a) & = & (v/c)/\sqrt{(}1 - v^2/c^2). \end{array} \tag{5.19}$$

Putting these into the transformations of $\hat{z}$ and $\hat{t}$, we see that:

$$\begin{array}{rcl} \hat{z} & \rightarrow & (\hat{z} + \hat{t}v/c)/\sqrt{(}1 - v^2/c^2), \\ \hat{t} & \rightarrow & (\hat{t} + \hat{z}v/c)/\sqrt{(}1 - v^2/c^2), \end{array} \tag{5.20}$$

which is the usual form for a boost in the $\hat{z}$ direction.

More generally, let $\vec{u} = (u_x, u_y, u_z)$ be a unit vector. Then the bivector

$$U_R = u_x\widehat{yz} - u_y\widehat{xz} + u_z\widehat{xy} \tag{5.21}$$

generates a rotation through the $\vec{u}$ axis. That is, to rotate by an angle θ around the $\vec{u}$ axis, one uses the transformation:

$$M \rightarrow e^{-\theta U/2}\, M\, e^{+\theta U/2}. \tag{5.22}$$

Similarly, to boost in a direction $\vec{v} = (v_x, v_y, v_z)$ one uses the bivector

$$V_B = v_x\widehat{xt} + v_y\widehat{yt} + v_z\widehat{zt}. \tag{5.23}$$

In general, boosts and rotations do not commute. However, let $\vec{v} = \vec{u}$, (or $= -\vec{u}$) and compute the product of the two generators:

$$U_R U_B = (u_x\widehat{yz} - u_y\widehat{xz} + u_z\widehat{xy})(u_x\widehat{xt} + u_y\widehat{yt} + u_z\widehat{zt}). \tag{5.24}$$

The above consists of 9 products. The off diagonal products, such as $u_x\widehat{yz}u_y\widehat{yt}$ and $u_y\widehat{yt}u_x\widehat{yz}$, come in pairs and like this pair, their unit multivectors anticommute and therefore the pairs all cancel. The only terms left are the diagonal terms, and therefore:

$$\begin{array}{rcl} U_R U_B & = & u_x^2\widehat{xyzt} + u_y^2\widehat{xyzt} + u_z^2\widehat{xyzt}, \\ & = & \widehat{xyzt}. \end{array} \tag{5.25}$$

The above is i times the helicity operator.

The rotation bivector U_R of Eq. (5.21) squares to -1. To convert it to a root of unity, we multiply it by i. On the other hand, the boost bivector, U_B, is already a root of unity. Since these two commute, they form a complete set of commuting roots of unity as we have used to define primitive idempotents. The helicity operator is the third non scalar element of the group generated by U_R and U_B. All accepted experimental observations are apparently compatible with the assumption that the laws of physics are symmetric under rotations and boosts. It is therefore very natural that helicity is so important to elementary particles.

5.2 Continuous Symmetries

Our primary consideration in this book are the elementary particles, and since we are associating the elementary particles with the primitive idempotents, it is natural for us to examine the exponential transformations from the point of view of what they do to the primitive idempotents.

Suppose that ι_χ and ι'_χ square to $+1$ and commute so that they can be used as a complete set of commuting roots of unity, which then define a complete set of primitive idempotents. Then these primitive idempotents are eigenvectors of ι_χ with eigenvalues of ± 1:

$$\begin{aligned} \iota_\chi 0.25(\hat{1} \pm \iota_\chi)(\hat{1} + \kappa\iota'_\chi) &= \pm 0.25(\hat{1} \pm \iota_\chi)(\hat{1} + \kappa\iota'_\chi), \\ \iota_\chi \rho_\pm &= \pm\rho_\pm, \end{aligned} \tag{5.26}$$

where κ is either $+1$ or -1, we don't care which. Then $\rho_\pm$ commutes with ι_χ and therefore this primitive idempotent is left unchanged by the exponential transformation generated by ι_χ:

$$\begin{aligned} \rho_\pm \quad\rightarrow\quad & e^{-a\iota_\chi}\rho_\pm e^{+a\iota_\chi}, \\ & = \rho_\pm e^{-a\iota_\chi} e^{+a\iota_\chi}, \\ & = \rho_\pm. \end{aligned} \tag{5.27}$$

Therefore, the primitive idempotents that can be made from a given root of unity are precisely the quantum states that are the fixed points of the exponential transformation defined by that root of unity.

If χ squares to -1, it is not a root of unity. In this case we cannot make primitive idempotents out of χ but we must instead use $i\iota_\chi = i\,\chi$. The above analysis will then apply. That is, the primitive idempotents made from ι_χ will be just those states that are unchanged by the exponential transformation defined by χ.

Before we examine the effect of the exponentials of the remaining unit multivectors, we should first discuss the classification of the Dirac unit multivectors. As with the Pauli unit multivectors, the Dirac unit multivectors can be classified according to the number of vectors, that is, by the blade. There are 5 blades:

$$\begin{gathered} \{\hat{1}\}, \\ \{\hat{x}, \hat{y}, \hat{z}, \hat{t}\}, \\ \{\widehat{xy}, \widehat{xz}, \widehat{yz}, \widehat{xt}, \widehat{yt}, \widehat{zt}\}, \\ \{\widehat{yzt}, \widehat{xzt}, \widehat{xyt}, \widehat{xyz}\}, \\ \{\widehat{xyzt}\}. \end{gathered} \tag{5.28}$$

This classification makes sense from a Clifford algebra sense, but not so much sense from a physics standpoint. It is better to treat $\hat{t}$ as different from the spatial vectors. This gives eight "time-blades":

$$\begin{array}{cc} \{\hat{1}\}, & \{\hat{t}\} \\ \{\hat{x}, \hat{y}, \hat{z}\}, & \{\widehat{xt}, \widehat{yt}, \widehat{zt}\}, \\ \{\widehat{xy}, \widehat{xz}, \widehat{yz}\}, & \{\widehat{yzt}, \widehat{xzt}, \widehat{xyt}\}, \\ \{\widehat{xyz}\}, & \{\widehat{xyzt}\}. \end{array} \tag{5.29}$$

The scalar, $\hat{1}$ generates only the trivial exponential transformation that doesn't move anything around, so we won't discuss it further. Of the remaining seven, they each square to $+1$ or -1. Following the mathematics of the previous section, the unit multivectors that square to $+1$ will generate transformations that looks like boosts, that is, that uses hyperbolic cosines and sines, while the unit multivectors that square to -1 will generate transformations that look like rotations. But we can always multiply by i to convert unit multivectors from the $+1$ square type to the -1 type or vice-versa so we need not concern ourselves much with this.

A more concise way of describing the effects of the exponential transformations is to describe their Lie algebras instead of their Lie groups. That is, we will look at the infinitesimal boosts and rotations instead of the boosts and rotations. Practically, this means that instead of writing, for the exponential transformation generated by $\widehat{xy}$ that $\hat{x} \to \cos(2a)\hat{x} + \sin(2a)\hat{y}$, we will instead simply write half the infinitesimal generator of the rotation, $+\hat{y}$.

We will list only the transformations for the unit multivectors that are neutrally oriented or are z oriented. The remaining unit multivectors are analogous to the z oriented ones. Then the complete set of infinitesimal rotation generators are:

	$\widehat{xy}$	$\widehat{zt}$	$\widehat{xyzt}$	$\hat{z}$	$\hat{t}$	$\widehat{xyz}$	$\widehat{xyt}$
$\hat{x}$	$+\hat{y}$	0	$+\widehat{yzt}$	$+\widehat{xz}$	$+\widehat{xt}$	0	0
$\hat{y}$	$-\hat{x}$	0	$-\widehat{xzt}$	$+\widehat{yz}$	$+\widehat{yt}$	0	0
$\hat{z}$	0	$+\hat{t}$	$+\widehat{xyt}$	0	$+\widehat{zt}$	0	$+\widehat{xy}$
$\hat{t}$	0	$+\hat{z}$	$+\widehat{xyz}$	$-\widehat{zt}$	0	$-\widehat{xyzt}$	0

(5.30)

The first column gives the rotation around the z-axis. The second column gives the boost in the z direction. The third column is helicity.

5.3 Velocity

In quantum mechanics, the operators for energy and momentum arise as generators of infinitesimal translations in time and position, respectively. In defining this, there is no need to include any geometric information as such, the same translations apply to all sorts of particles of whatever spin. As such, there is no geometric content of the Clifford algebra sort involved in their definitions. In addition, energy and momentum have units that involve mass, and since the origin of mass is a primary topic of this book, were we to begin by assuming energy and momentum we would be making an argument that is arguably circular. Max Jammer discusses this problem at length in [16].

On the other hand, position and velocity are concepts that do not require mass to be defined. The concepts of position and time arise naturally from our coordinate system, they are simply (x, y, z) and t. And it turns out that velocity is also a natural geometric object in the Clifford algebra theory, which we now discuss. From here on, we will be working with the massless Dirac equation instead of the massive one.

Let $\Psi(x,y,z,t)$ be a plane wave that is moving with velocity $\vec{v} = (v_x, v_y, v_z)$. A plane wave has the same value across the surface of a plane so that the value of the wave can be completely determined for all positions and time just by watching what values occur at a particular point over all time. The speed of the plane wave is $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$. The part of this speed in the x direction is v_x/v. So the point $(x,0,0)$ gets its values delayed by $x(v_x/v)/v$ from the value of the plane wave at $(0,0,0)$. Similarly for the point $(0,y,0)$ and $(0,0,z)$. Accordingly, we can write:

$$\Psi(x,y,z,t) = \Psi(0,0,0,t - (xv_x + y/_y + zv_z)/v^2). \tag{5.31}$$

We assume that Ψ satisfies the generalized *massless* Dirac equation. This allows us to compute as follows:

$$\begin{aligned} 0 &= (\nabla)\Psi(x,y,z,t), \\ &= (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{t}\partial_t)\Psi(0,0,0,t - (xv_x + yv_y + zv_z)/v^2), \\ &= (\hat{t} - (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})/v^2)\Psi_t(0,0,0,t). \end{aligned} \tag{5.32}$$

To put the above into a form more familiar from the point of view of primitive idempotents, we multiply on the left by $-0.5\hat{t}$. We obtain:

$$0 = 0.5(\hat{1} - (v_x\widehat{xt} + v_y\widehat{yt} + v_z\widehat{zt})/v^2)\Psi_t(0,0,0,t). \tag{5.33}$$

Of course $\Psi(0,0,0,t) =$ constant solves the above. But if we want more interesting plane waves, that is, if Ψ_t is not everywhere zero, we need to have Ψ_t multiply against $0.5(\hat{1} - (\widehat{xt}v_x + \widehat{yt}v_y + \widehat{zt}v_z)/v^2)$ to give zero.

Multiplying to give zero is what annihilating idempotents are for. Idempotents are defined by roots of unity, so we look for one in this problem. Note that

$$(\widehat{xt}v_x + \widehat{yt}v_y + \widehat{zt}v_z)^2 = (v_x^2 + v_y^2 + v_z^2)\hat{1}, \tag{5.34}$$

so $\iota_v = (\widehat{xt}v_x + \widehat{yt}v_y + \widehat{zt}v_z)/v^2$ is a root of unity, if $v = 1$, and defines annihilating idempotents:

$$0.5(\hat{1} \pm \iota_v) = 0.5(\hat{1} \pm (\widehat{xt}v_x + \widehat{yt}v_y + \widehat{zt}v_z)). \tag{5.35}$$

Comparing with Eq. (5.33) we see that Ψ_t will be annihilated if we put:

$$\Psi_t = 0.5(\hat{1} + \widehat{xt}v_x + \widehat{yt}v_y + \widehat{zt}v_z)dK/dt, \tag{5.36}$$

where $K(t)$ is an arbitrary[2] Clifford algebra constant times any scalar function of time. An example function for K that has the usual sinusoidal time dependence is $K(t) = (\widehat{xyz} + 3\hat{x} - 12)\sin(t)$.

What we have demonstrated here is that we can produce velocity eigenstates of the massless Dirac equation. In the previous section we showed that the exponential transformations produced from $\widehat{xt}$, $\widehat{yt}$ or $\widehat{zt}$ corresponded to boosts

[2]Okay, one that does not annihilate $0.5(1 + \iota_v)$.

in the directions given by the orientation of these unit multivectors. Now we see that the velocity eigenstates of the massless Dirac equation are the states that are eigenstates of boosts.

Furthermore, in examining the chiral representation of the Dirac equation in Section (4.5), we saw that the three commuting roots of unity chosen to define the diagonal primitive idempotents of the representation were the operator for spin in the z direction $\widehat{ixy}$, the helicity operator $\widehat{ixyzt}$, and what we now see to be the operator for velocity in the z direction: $\widehat{zt}$.

That the Dirac equation produces natural velocity eigenstates, with velocity eigenvalues of ± 1, has been known since the equation was first discovered. The idea was explored in the "zitterbewegung" theory of the electron. This theory supposed that the electron exhibited very high speed motion at speed c, and an approximately stationary electron is produced only by the high frequency reversals of direction. This theory has been explored by David Hestenes using the geometric algebra in the context of the state vector formulation of quantum mechanics.[17] This book will reopen this idea, but from a density operator point of view, and with chiral states.

5.4 Helicity and Proper Time

This section is rather speculative and is not entirely necessary, but is included for completeness. The helicity operator, $\gamma^5 = -\widehat{ixyzt}$ has some interesting properties. It squares to $+1$ and it anticommutes with the whole set of basis vectors:

$$\begin{aligned}
(-\widehat{ixyzt})^2 &= 1, \\
\hat{x}\,(-\widehat{ixyzt}) &= -(-\widehat{ixyzt})\,\hat{x}, \\
\hat{y}\,(-\widehat{ixyzt}) &= -(-\widehat{ixyzt})\,\hat{y}, \\
\hat{z}\,(-\widehat{ixyzt}) &= -(-\widehat{ixyzt})\,\hat{z}, \\
\hat{t}\,(-\widehat{ixyzt}) &= -(-\widehat{ixyzt})\,\hat{t}.
\end{aligned} \tag{5.37}$$

These are exactly the attributes that we would require of a spatial basis vector. For example, compare the above with the following equations where $-\widehat{ixyzt}$ is replaced by $\hat{z}$:

$$\begin{aligned}
(\hat{z})^2 &= 1, \\
\hat{x}\,(\hat{z}) &= -(\hat{z})\,\hat{x}, \\
\hat{y}\,(\hat{z}) &= -(\hat{z})\,\hat{y}, \\
\hat{t}\,(\hat{z}) &= -(\hat{z})\,\hat{t}.
\end{aligned} \tag{5.38}$$

We are therefore somewhat justified in thinking of $-\widehat{ixyzt}$ as an additional spatial basis vector and write:

$$\hat{s} = -\widehat{ixyzt}. \tag{5.39}$$

There is a problem; "what direction should we associate with $\hat{s}$?",but ignoring that for the moment, the above switch replaces our $4-$dimensional complex Clifford algebra generated by the 4 basis vectors $\{\hat{x}, \hat{y}, \hat{z}, \hat{t}\}$ with a

5−dimensional real Clifford algebra generated by the 5 basis vectors $\{\hat{x}, \hat{y}, \hat{z}, \hat{s}, \hat{t}\}$. In this section we explore this version of the Dirac algebra.

In this new 5−dimensional Clifford algebra, the imaginary unit becomes a real geometric object:

$$\begin{aligned} \hat{s} &= -\widehat{ixyzt}, \\ \hat{i}\hat{s}\hat{s} &= -\hat{i}\hat{s}\widehat{ixyzt}, \\ \hat{i} &= +\hat{s}\,\widehat{xyzt}, \\ &= -\widehat{xyzst}. \end{aligned} \tag{5.40}$$

Thus the usual complex Dirac algebra can be thought of as a real 5−dimensional Clifford algebra. But if we are to do more than just think of $\hat{s}$as a spatial vector, we need to associate some sort of spatial coordinate with the basis vector.

The original reason Dirac designed the gamma matrices was to achieve a relativistic quantum mechanics, so it is natural to return to relativity for an inspiration on how to interpret $\hat{s}$. The signature of the Dirac algebra comes from the line element, which we will choose to write as:

$$\begin{aligned} -ds^2 &= -dt^2 + (dx^2 + dy^2 + dz^2), \\ &= (\hat{t}dt)^2 + (\hat{x}dx)^2 + (\hat{y}dy)^2 + (\hat{z}dz)^2. \end{aligned} \tag{5.41}$$

The above is a bit ugly from a geometric point of view because the right hand side is written as an operator, but the left hand side is a scalar. To treat ds as if it were a differential associated with a coordinate, we move it to the right hand side and bring in the basis vector $\hat{s}$.The above line element is transformed to

$$0 = (\hat{x}dx)^2 + (\hat{y}dy)^2 + (\hat{z}dz)^2 + (\hat{s}ds)^2 + (\hat{t}dt)^2. \tag{5.42}$$

The new line element describes a spacetime with four spatial dimensions and one time dimension, but there is no longer a scalar proper time on the left hand side. This would be appropriate for a theory that applied to massless particles that all travel at the speed of light in the presence of a hidden dimension with a circular coordinate s.

Similar ideas appear in Jose Almeida's work[18], which rewrites gravitational dynamics, electromagnetism, and quantum mechanics in a common mathematical approach using Clifford / Geometric algebra by supposing "that all material particles follow null geodesics of a 5D space". He uses the same signature, $-++++$ for the five dimensions as here. The primary difference is that he works in the state vector formalism instead of the density operator formalism. A less important difference is that he does not choose to assume that $\hat{s}$ corresponds to a literal hidden dimension.

It should be mentioned that if we were to choose the opposite signature for a 5−dimensional Clifford algebra, $+----$, there would be no geometric imaginary unit. Instead, while $\widehat{xyzst}$ would still commute with the algebra, it would square to $+1$ and we would be faced with the predicament of having a commuting pseudoscalar 1 different from the usual one.

Chapter 6

Measurement

No proposition Euclid wrote,
    No formulae the text-books know,
Will turn the bullet from your coat,
    Or ward the tulwar's downward blow.
Strike hard who cares--shoot straight who can—
The odds are on the cheaper man.

WITHOUT MEASUREMENT, waves need never be associated with particles. Measurement is the heart of quantum mechanics, the difference between it and classical mechanics.

In the 1950s, Julian Schwinger wrote down an elegant but little noticed foundation for quantum mechanics that he called the "Measurement Algebra". In this chapter we introduce what is now known as "Schwinger's Measurement Algebra", and apply the primitive idempotent calculations of the previous chapters to the problem of determining the structure of the natural elementary particles of a Clifford algebra.

6.1 The Stern-Gerlach Experiment

An experiment, simple in concept, that illustrates the difference between classical and quantum effects is that originally carried out by Otto Stern and Walter Gerlach in 1922. An oven heats up atoms, for example silver, until they form a gas. In a vacuum, the silver atoms travel in straight lines and can be formed into a beam by a plate with a small hole in it. The beam of atoms then travels through the influence of a magnetic field to a screen. See Fig. (6.1).

Classically, the atoms act like small magnets. The reader is familiar with how small magnets act in the presence of a magnetic field because this is the principle on which the magnetic compass operates. The magnetic field induces a torsion on the compass magnet, causing the compass to turn until it aligns itself with the local magnetic field. But there is no net attractive force on the magnet.

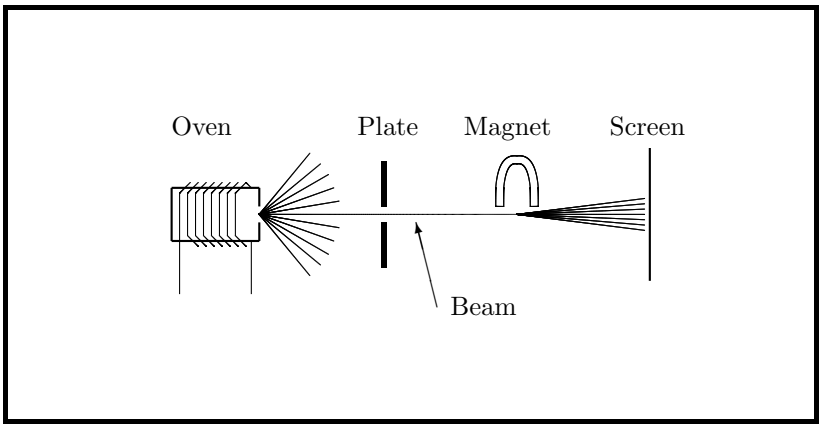


Figure 6.1: The Stern-Gerlach experiment: Atoms are heated to a gas in an oven. Escaping atoms are formed into a beam by a small hole in a plate. A magnet influences the beam, which then forms a figure on a screen.

On the other hand, the reader is also familiar with the fact that a small magnet can feel not just torsion, but attraction and repulsion when it is in the presence of the magnetic field created by another small magnet. The difference between this case and the compass is that Earth's magnetic field is locally homogenous. To arrange for a small magnet to feel attraction and repulsion, one must arrange for the magnetic field to be non homogenous.

So Stern and Gerlach ran the beam of atoms through a magnetic field that was as non homogeneous as they could create. They expected that the various atoms would feel different forces depending on their orientation and that this would spread out the pattern deposited on the screen.

We can only imagine their surprise and consternation when the figure turned out not to be simply an elongated spot, but instead split into two distinct spots. The usual explanation for an unexpected result in an experiment such as this, is inattentiveness on the part of a graduate student, but the effect was real. An atom either went up or down, it was as if there were only two possible orientations for the atom's magnetic field, "up" or "down".

The tendency to go up or down in an inhomogeneous magnetic field turned out to be preserved. If one adjusted the screen to block only the atoms in the "down" spot, leaving the "up" atoms as a beam, and arranged for another inhomogeneous magnetic field for these atoms to traverse, all of these atoms again took the "up" path. Of course "up" and "down" are determined by the orientation of the magnet, for example, the magnets might be arranged so that their north poles point up and south poles down.

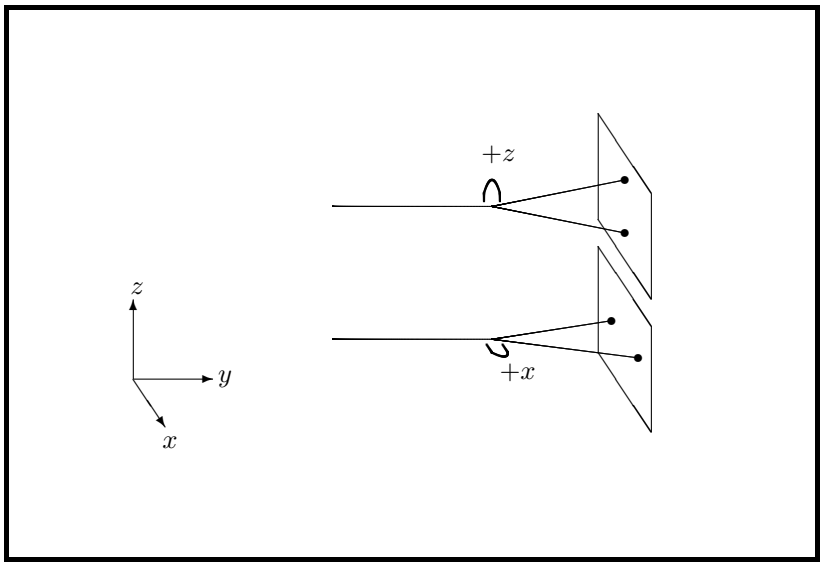


Figure 6.2: Two Stern-Gerlach experiments, one oriented in the $+z$ direction, the other oriented in the $+x$ direction.

There are two orientations involved in the Stern-Gerlach experiment. The experiment requires an inhomogeneous magnetic field, and the orientation of this field determines how the beam is split. This is the obvious, macroscopic orientation. The second orientation is the orientation of the magnetic field of the atoms. This orientation is microscopic and our understanding of it is only indirect.

We can imagine arranging the magnetic field in orientations other than up and down. If the magnet is oriented horizontally instead of vertically, the beam is deflected into two horizontal spots instead of vertical, see Fig. (6.2). This raises the question of what happens when the Stern-Gerlach experiment is performed on a beam of atoms that have previously been separated with a different orientation.

It turns out that the results of this sort of experiment can be accurately predicted using the projection operators of the Pauli algebra. The reason for this is that the two spots are evidence of a spin-1/2 quantum state, just the states of the Pauli algebra. When other atoms are sent through a Stern-Gerlach experiment, the number of spots they produce can be different from two. But the elementary fermions are spin-1/2 particles and the case of two spots is the one we will examine in great detail.

6.2 Filters and Beams

Operators as used in this book, have one input and one output. On the other hand, the Stern-Gerlach experiment has one input and two outputs. To make the Stern-Gerlach experiment fit into our mathematical apparatus, we have to rearrange it so that it has only one output. The way to do this is to imagine that after the beam is split in two, only one of the beams is allowed to exit the experiment. The other is absorbed by a beam stop. We will call this type of apparatus a "Stern-Gerlach Filter".

We want to model the effect of sending the output beam of a Stern-Gerlach filter directly into the input of another Stern-Gerlach filter so we must ignore the details of the oven that produced the atoms, the plate that defined the beam, and the screen that absorbed the spots. Instead, we will pay attention only to the inhomogeneous magnetic field. More, we will ignore the details of the magnetic field and pay attention only to its orientation.

The inhomogeneous magnetic field of a Stern-Gerlach experiment is arranged to cause the atoms to split in two beams going in slightly different directions. Since a Stern-Gerlach filter keeps only one of these two beams, we can define the orientation of the filter by the direction that is kept. For example, if a horizontal beam is split by the experiment into two beams, one going slightly up and the other going slightly down, and the filter keeps the beam going slightly up, then the orientation we associate with this filter will be "up", which we designate by a unit vector, $(0, 0, 1)$

Let $\vec{v} = (v_x, v_y, v_z)$ be a unit vector. The projection operator for spin in the $\vec{v}$ direction is:

$$\rho_v = 0.5(1 + v_x\hat{x} + v_y\hat{y} + v_z\hat{z}). \tag{6.1}$$

This projection operator satisfies the idempotency relation:

$$\rho_v \; \rho_v = \rho_v, \tag{6.2}$$

and this matches the experimentally observed fact that a beam that passes a Stern-Gerlach filters oriented in the $\vec{v}$ direction will then pass another Stern-Gerlach filter oriented in that same direction.

Given a Stern-Gerlach filter oriented in the direction $\vec{v}$, we can imagine the complementary Stern-Gerlach filter that switches the beam stop between the two split beams. Instead of passing the up beam and halting the down beam, the complementary Stern-Gerlach filter does the reverse. The projection operator associated with this complementary Stern-Gerlach filter is the projection operator associated with the vector in the opposite direction.

The Stern-Gerlach apparatus oriented in the $+z$ direction is represented by the projection operator:

$$\rho_{+z} = 0.5(1 + \hat{z}), \tag{6.3}$$

and so the complementary projection operator is

$$\rho_{-z} = 0.5(1 - \hat{z}). \tag{6.4}$$

These two projection operators annihilate:

$$\rho_{+z}\ \rho_{-z} = 0, \tag{6.5}$$

and this corresponds to the fact that two consecutive Stern-Gerlach filters oriented in opposite directions will pass no beam – they act as a beam stop. Since the associated projection operators multiply to give zero, we associate zero with a beam stop.

Since we are representing consecutive experiments by multiplication, the experiment that does nothing to the beam must be represented by $\hat{1}$, as "doing nothing" must commute with all other possible experiments. On the other hand, the sum of two complementary projection operators is unity. For example:

$$\rho_{+z} + \rho_{-z} = 0.5(1 + \hat{z}) + 0.5(1 - \hat{z}) = \hat{1}. \tag{6.6}$$

This meets up with the experimental fact that one can split a beam with a beam splitter, recombine the two split beams, and the result is a beam indistinguishable from the original.

6.3 Statistical Beam Mixtures

In the operator formalism, one represents quantum states as operators. With the Stern-Gerlach filters also represented by operators, the mathematical language is unified and simplified. But the operators that represent Stern-Gerlach filters are projection operators while quantum states are more general.

Let ρ_{+z} and ρ_{-z} be the projection operators that represent Stern-Gerlach filters oriented in the $+z$ and $-z$ direction. While we have been using these operators to represent the Stern-Gerlach filters, we can also think of them as representing the beam that results when an unpolarized beam is sent through their respective Stern-Gerlach filter. This is an important philosophical point– we are using the same mathematical object to represent both the beam and the filter that produces the beam. But we can only do this when the intensity of the beam is known, and this means that we are assuming implicitly some unit intensity for the unpolarized beam applied to the filter.

In recombining two beams, we earlier noted that $\rho_{+z} + \rho_{-z} = \hat{1}$, that is, the act of splitting and recombining is equivalent to leaving the beam unmodified. Our use of the notation here is very natural from the point of view of the operators but it is at variance with the traditional way of doing quantum statistical mechanics. The difference amounts to a choice of normalization.

In the standard formalism of quantum mechanics, the normalization of quantum states is defined so that probabilities are easy to compute. This book abandons this tradition and instead normalizes quantum states so that their energies are defined naturally. This is why we can split and recombine beams and have their quantum states computed by simple addition; In our formalism when one represents a beam by an operator, the scalar part of the operator defines the beam intensity in terms of particles per second or what have you. When an unpolarized beam is divided in two by a Stern-Gerlach experiment, our intensity is divided as well.

To illustrate the standard method of describing a mixed beam, let us consider making a beam that is partly spin up and partly spin down. Since the standard method normalizes probabilities, we have to choose two fractions that add to unity. For example, we can have a beam that is p spin up and q spin down where $p + q = 1$, and represent the combined beam, ρ by:

$$\begin{aligned} \rho &= p\rho_{+z} + q\rho_{-z}, \\ &= 0.5p + 0.5p\hat{z} + 0.5q - 0.5q\hat{z}, \\ &= 0.5 + 0.5(p-q)\hat{z}, \\ &= \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \end{aligned} \tag{6.7}$$

The traditional quantum statistical entropy of a mixture is defined by:

$$S = -k \ \mathrm{tr}(\rho \ \ln(\rho)), \tag{6.8}$$

where k is Boltzmann's constant. For a diagonalized state, the logarithm of the matrix is easy to compute and one finds:

$$\begin{aligned} S_\rho &= -k(p\ln(p) + q\ln(q)), \\ &= -k\ln(p^p q^q). \end{aligned} \tag{6.9}$$

In the above, note that since p and q are between 0 and 1, so are p^p and q^q. The entropy is therefore positive. The maximum value for the entropy occurs when $p = q = 0.5$ and is $k\ln(2)$.

In classical thermodynamics, whenever two separated systems are brought into contact, their total entropy is expected to increase or, at best, stay constant. Therefore the quantum states that maximize entropy are of particular interest.

Suppose that we have a complete set of N mutually annihilating primitive idempotents. We can imagine having a beam made up of a mixture of p_1 of the first stateρ_1, p_2, of the second state ρ_2, and so on up to p_N of the Nth stateρ_N, with $p_1 + p_2 + ... + p_N = 1$. Then the entropy is defined by

$$S = -k\ln(p_1^{p_1} p_2^{p_2} ... p_N^{p_N}). \tag{6.10}$$

A small amount of calculus shows that this function is maximized when $p_1 = p_2 = ... = p_N = 1/N$.

Since $\rho_1, ...\rho_N$ form a complete set of primitive idempotents, they sum to unity. Therefore, the quantum mixture that maximizes entropy can be represented by the operator:

$$\rho_{\text{Max}} = (\rho_1 + \rho_2 + ... + \rho_N)/N = \hat{1}/N. \tag{6.11}$$

Thus we see that the quantum mixture that has the maximum entropy happens to be the mixture that is purely scalar. Furthermore, for small deviations from this maximum entropy mixture, the entropy is quadratic in the non scalar amounts. In a later chapter, when we discuss the natural force between primitive idempotents, we will revisit these observations.

6.4 Schwinger's Measurement Algebra

In discussing the projection operator that picks out a quantum state of type a, our notation has been to write ρ_a. In our notation, this describes both the fields of the Stern-Gerlach type experiment that allows only that quantum state to pass, and and also to describe the quantum state itself. In Schwinger's notation,[9] we would write this same object as $m(a)$. The choice of notation is not significant, and to reduce confusion we will stick with the density operator notation except for these few paragraphs that will place the present effort in historical context.

In Schwinger's exposition, he quickly passes from the projection operators $m(a)$ to symbols that represent Stern-Gerlach filters that alter the beam. For example, one could arrange for a filter that accepts only spin oriented in the $+z$ direction, and releases particles with spin oriented in the $+x$ direction. Schwinger would write such a filter with a more general notation as $m(+x, +z)$. In this, Schwinger's notation exceeds what ours is capable of.

Schwinger measurement symbols of the type $m(a, b)$ incorporate a disturbance to the quantum state. We reject these for two reasons. First, we want physical processes to occur when our quantum states are disturbed. Second, measurement symbols of this type automatically imply an arbitrary choice of phase.

So long as the two quantum states a and b do not annihilate each other, we can uniquely define the Schwinger measurement symbol $m(a, b)$ as the product of the operators:

$$m(a, b) = \rho_a \, \rho_b. \tag{6.12}$$

But if the quantum states annihilate each other, the above definition fails. For example, $\rho_{+z}\rho_{-z} = 0$.

We could avoid the annihilation problem by choosing an intermediate state. For example, we could write:

$$m(+z, -z) = \rho_{+z} \, \rho_{+x} \, \rho_{-z}, \tag{6.13}$$

but the result will be different depending on the choice of that intermediate state. For example,

$$\begin{array}{rcl} \rho_{+z}\rho_{+x}\rho_{-z} & = & +0.25(\hat{x}+\widehat{iy}), \quad \text{but} \\ \rho_{+z}\rho_{-x}\rho_{-z} & = & -0.25(\hat{x}+\widehat{iy}). \end{array} \tag{6.14}$$

In fact we can get any arbitrary phase by judicious choice of the intermediate state. And furthermore, since spinors carry arbitrary phases, each of these objects will convert a $-z$ ket to a $+z$ ket.

If the two states, a, and b are chosen at random, it is highly unlikely (i.e. a set of measure zero) that they will annihilate. Far more likely is that $m(a,b)$ will be well defined by $\rho_a\ \rho_b$. From this one might suppose that one could use the general measurement symbols with little likelihood of problem. However, the general measurement symbols between annihilating states are precisely the measurement symbols that the standard way of formulating quantum mechanics finds most useful–they are the raising and lowering operators. Our inability to define their phases in a unique way is equivalent to the fact that when one defines Clebsch-Gordon coefficients one must choose arbitrary complex phases.

The Schwinger notation can be made uniquely defined by assuming a vacuum state ρ_0 that does not annihilate any quantum state. And this is how Schwinger derives the usual state vector formalism for quantum mechanics from his measurement algebra. But this book is devoted to the operator formalism, as opposed to the spinor formalism and we will not assume what Schwinger refers to in [9] as a "fictitious" vacuum state. This ends our historical digression, and we now return to the density operator, ρ_χ, notation.

Given that any physical quantity A (for example, spin) assumes only a finite number of distinct values, a_1, a_2, ... a_n, the following rules apply:

$$\begin{array}{rcl} \rho_{aj}\ \rho_{aj} & = & \rho_{aj}, \\ \rho_{aj}\ \rho_{ak} & = & 0, \quad \text{if } j \neq k, \\ \sum_j \rho_{aj} = 1. & & \end{array} \tag{6.15}$$

The above are identical to the rules that govern a complete set of primitive idempotents of a Clifford algebra so it is natural that we will use Clifford algebra to analyze the elementary measurements of Schwinger's measurement algebra.

So long as we restrict the physical quantity A to be just spin we have not gained anything above what we already know. But we can interpret the physical quantity A as including more than just spin. A could represent any other complete set of quantum states, where "complete" depends on our context.

We've introduced primitive idempotents in a bottom up manner. That is, we begin with a Clifford algebra, define a complete set of commuting roots of unity, and from that define a complete set of primitive idempotents.

Schwinger's measurement algebra affords a method of defining these objects from the top down.

For example, if we are concerned with solutions to the Dirac equation for the electron, we could choose that A includes information about a state's spin and charge: $\{+ze, -ze, +z\bar{e}, -z\bar{e}\}$. With the choice of these four states, the primitive idempotents would be:

$$\rho_{+ze}, \rho_{-ze}, \rho_{+z\bar{e}}, \rho_{-z\bar{e}}. \tag{6.16}$$

In most quantum mechanics books, the operator for spin$-1/2$ in the $\pm z$ direction squares to $1/4$, but we will instead define the spin$-1/2$ operator as twice the usual definition so it will instead square to 1. This makes S_{+z} a "root of unity" that we can use to define primitive idempotents, and we can define the projection operator for spin$-1/2$ in the $\pm z$ directions as

$$\rho_{\pm z} = 0.5(1 + S_{+z}). \tag{6.17}$$

The above two equations can be solved to give S_{+z} in terms of $\rho_{\pm z}$:

$$S_{+z} = \rho_{+z} - \rho_{-z}. \tag{6.18}$$

Thus we can derive the root of unity from the projection operators of the Schwinger measurement algebra.

With the choice of primitive idempotents in Eq. (6.16), we can define the projection operator for spin$-1/2$ in the $\pm z$ direction as:

$$\begin{array}{rcl} \rho_{+z} & = & \rho_{+ze} + \rho_{+z\bar{e}}, \\ \rho_{-z} & = & \rho_{-ze} + \rho_{-z\bar{e}}. \end{array} \tag{6.19}$$

Applying Eq. (6.18), we derive:

$$S_{+z} = (\rho_{+ze} + \rho_{+z\bar{e}}) - (\rho_{-ze} + \rho_{-z\bar{e}}). \tag{6.20}$$

The above spin operator, S_{+z} is now defined entirely within the Schwinger measurement algebra. The rule for obtaining primitive roots of unity from the Schwinger measurement algebra is to simply add all the eigenstates with eigenvalue $+1$, and subtract the state with eigenvalue -1.

Note that when an operator is defined as above, that is, as a sum over a complete set of states each independently multiplied by ± 1, the operator is automatically a root of unity. When squared, the cross terms will all be zero as different primitive idempotents annihilate, and the squared terms will convert the $-$ signs to $+$. Then this will be a sum over a complete set of primitive idempotents and so is $\hat{1}$.

Thus we have ways of moving back and forth between a Schwinger measurement algebraic description of a set of quantum states (i.e. a list of sets of observed quantum numbers), and a description of a set of quantum states that are primitive idempotents of a Clifford algebra. In later chapters we will apply this technology to the problem of uniquely defining the elementary particles as composites of primitive idempotents of a Clifford algebra.

6.5 Clifford Algebra and the SMA

As we add vectors to a Clifford algebra, we get larger numbers of commuting roots of unity in our complete sets of commuting roots of unity. If there is only one vector, say $\hat{x}$, then there is only one root of unity, $\hat{x}$. In this case, there are only 2 primitive idempotents, $0.5(1 \pm \hat{x})$.

With two vectors, $\{\hat{x}, \hat{y}\}$, we can still only define a single commuting root of unity. If we choose $\hat{x}$ the remaining degrees of freedom in the Clifford algebra, namely $\{\hat{y}, \widehat{xy}\}$, all anticommute with $\hat{x}$ and so we are stuck with only one commuting root of unity. In this situation, as in the case with only one vector, there are only 2 primitive idempotents, $0.5(1 \pm \hat{x})$.

The Pauli algebra has three vectors, $\{\hat{x}, \hat{y}, \hat{z}\}$. If we choose $\hat{x}$ as a commuting root of unity, the remaining degrees of freedom are $\{\hat{y}, \hat{z}, \widehat{yz}, \widehat{xy}, \widehat{xz}, \widehat{yz}, \widehat{xyz}\}$. Of these, $\widehat{yz}$ and $\widehat{xyz}$ commute with $\hat{x}$, but neither of these squares to -1. Therefore the (real) Pauli algebra defines only two primitive idempotents, $0.5(1 \pm \hat{x})$.

On the other hand, if we define a Clifford algebra with the three vectors $\{\hat{x}, \hat{y}, \hat{t}\}$, we can define two commuting roots of unity, for example, $\hat{x}$ and $\widehat{yt}$, and we would have four primitive idempotents, $0.25(1 \pm \hat{x})(1 \pm \widehat{yt})$. So we see that the number of primitive idempotents that are contained in a Clifford algebra can depend on its signature.

The Dirac algebra, with vectors $\{\hat{x}, \hat{y}, \hat{z}, \hat{t}\}$ also allows two commuting roots of unity, for example, $\hat{x}$ and $\widehat{zt}$, and therefore four primitive idempotents, $0.25(1 \pm \hat{x})(1 \pm \widehat{zt})$.

If we add another spatial vector, $\hat{s}$, to the Dirac algebra, to get the "proper time algebra" as was discussed in Sec. (5.4), we can still only write two commuting roots of unity, the same as with the Dirac algebra. To get three commuting roots of unity we could instead add a second time vector to the Dirac algebra, $\widehat{t'}$ with $(\widehat{t'})^2 = -1$. Then three commuting roots of unity would be $\{\hat{x}, \widehat{yt}, \widehat{zt'}\}$, and we would have eight primitive idempotents.

Another way to get three commuting roots of unity would be to take the proper time algebra to be complex rather than real. Then we could have three commuting roots of unity as $\{\hat{x}, \widehat{yt}, \widehat{izs}\}$ again giving eight primitive idempotents, that is, $0.125(1 \pm \hat{x})(1 \pm \widehat{yt})(1 \pm \widehat{izs})$. If $\hat{s}$ is treated as belonging to a compact or hidden dimension, then one can consider the complex numbers as indicating position in the s dimension. But in doing this, one ends up with two natural geometric imaginary units, i and $\widehat{xyzst}$, both of which square to -1 and commute with the geometry.

In any case, if we find M commuting roots of unity, we find 2^M primitive idempotents. Thinking of the M commuting roots of unity as operators, the 2^M primitive idempotents are eigenstates with eigenvalues of ± 1. For example:

$$\hat{x} \;\; 0.25(1 - \hat{x})(1 + \widehat{yt}) = -0.25(1 - \hat{x})(1 + \widehat{yt}), \tag{6.21}$$

so $0.25(1-\hat{x})(1+\widehat{yt})$ is an eigenstate of $\hat{x}$ with eigenvalue -1.

But the M roots of unity commute, so a primitive idempotent generated by these roots is also an eigenstate of any product of the M commuting roots of unity. So rather than having the primitive idempotents be eigenstates of M commuting independent operators, they are more generally eigenstates of 2^M commuting not independent operators. This confusing but simple concept is best illustrated with an example. The Dirac algebra is just a little too simple to get the point across, so instead we will use the complex proper time algebra.

Let our commuting roots of unity be $\{\hat{x}, \widehat{yt}, \widehat{izs}\}$. A particular primitive idempotent defined by these roots is:

$$\begin{aligned} \rho_{-+-} &= 0.125(1-\hat{x})(1+\widehat{yt})(1-\widehat{izs}), \\ &= 0.125(\hat{1}-\hat{x}+\widehat{yt}-\widehat{izs}-\widehat{xyt}+\widehat{ixzs}-\widehat{iyzst}+\widehat{ixyzst}). \end{aligned} \tag{6.22}$$

The set of all products of the three commuting roots of unity is the set with eight elements given by $\{\hat{1}, \hat{x}, \widehat{yt}, \widehat{xyt}, \widehat{izs}, \widehat{ixzs}, \widehat{iyzst}, \widehat{ixyzst}\}$. Note that these are precisely the degrees of freedom present in the expansion of ρ_{-+-} above, Eq. (6.22).

As each degree of freedom is represented in Eq. (6.22) with a plus or minus sign, so the primitive idempotent ρ_{-+-} is an eigenstate of that degree of freedom with eigenvalue $+1$ or -1. Using explicit multiplication, the reader is invited to verify that ρ_{-+-} is an eigenvalue of these eight commuting roots of unity with eigenvalues ± 1 where the sign is given by the sign in the expansion of ρ_{-+-}.

Using this principle, one can write down all the eigenvalues of all eight primitive idempotents $\rho_{\pm\pm\pm}$:

	$\hat{1}$	$\hat{x}$	$\widehat{yt}$	$\widehat{izs}$	$\widehat{xyt}$	$\widehat{ixzs}$	$\widehat{iyzst}$	$\widehat{ixyzst}$
ρ_{---}	$+1$	-1	-1	-1	$+1$	$+1$	$+1$	-1
ρ_{--+}	$+1$	-1	-1	$+1$	$+1$	-1	-1	$+1$
ρ_{-+-}	$+1$	-1	$+1$	-1	-1	$+1$	-1	$+1$
ρ_{-++}	$+1$	-1	$+1$	$+1$	-1	-1	$+1$	-1
ρ_{+--}	$+1$	$+1$	-1	-1	-1	-1	$+1$	$+1$
ρ_{+-+}	$+1$	$+1$	-1	$+1$	-1	$+1$	-1	-1
ρ_{++-}	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1
ρ_{+++}	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$

(6.23)

In the above, it should be noted that the signs are somewhat arbitrary in that $-\hat{x}$ makes just as good a root of unity as $+\hat{x}$. For this reason, not much can be made of the fact that the last row has all $+1$ eigenvalues.

One of the 2^M commuting non independent roots of unity is $\hat{1}$, and of course has trivial eigenvalues of $+1$ for all primitive idempotents. For the purpose of classifying quantum states we can ignore this operator. That leaves $2^M - 1$ good quantum numbers. Each of the other commuting roots of unity has equal numbers of eigenstates with $+1$ and -1 eigenvalues. Therefore one has that the sum of the quantum numbers over all states is zero, a fact that is reminiscent of the requirement for anomaly cancelation in QFT, but is beyond the scope of this book.

Suppose that we have three commuting roots of unity, ι_1, ι_2, and ι_3. The eight primitive idempotents are $0.125(1 \pm \iota_1)(1 \pm \iota_2)(1 \pm \iota_3)$. Any of these eight primitive idempotents has good eigenvalues with respect to any of the eight operators in the Abelian group generated by the three commuting roots of unity. The table of eigenvalues will be identical to Eq. (6.23) but we repeat it here:

	$\hat{1}$	ι_3	ι_2	ι_1	$\iota_1\iota_2$	$\iota_1\iota_3$	$\iota_2\iota_3$	$\iota_1\iota_2\iota_3$
ρ_{---}	+	−	−	−	+	+	+	−
ρ_{--+}	+	−	−	+	−	−	+	+
ρ_{-+-}	+	−	+	−	−	+	−	+
ρ_{-++}	+	−	+	+	+	−	−	−
ρ_{+--}	+	+	−	−	+	−	−	+
ρ_{+-+}	+	+	−	+	−	+	−	−
ρ_{++-}	+	+	+	−	−	−	+	−
ρ_{+++}	+	+	+	+	+	+	+	+

(6.24)

written in terms of three generic commuting roots of unity and the eight generic primitive idempotents they produce.

In standard quantum mechanics, different particles share the same Dirac equation without any way of distinguishing between them except in the notation of the physicist. In the standard model, one must anticommute identical fermion creation operators and the same with identical annihilation operators. This anticommutation enforces the Pauli exclusion principle. However, if two fermions are not identical, one commutes their creation and annihilation operators and there is no Pauli exclusion principle. In the standard formulation, one keeps track of which fermions are identical or non identical by notation.

In our formulation of quantum mechanics different primitive idempotents are distinguished not merely by notation, but by having different geometric values. For this reason, we do not need to rely on notation to keep track of whether our fermions are identical or not.

Suppose that we have two distinct primitive idempotents in the same complete set of primitive idempotents. These two idempotents annihilate, so their sum is also idempotent:

$$\begin{aligned} (\rho_1 + \rho_2)^2 &= (\rho_1)^2 + \rho_1\rho_2 + \rho_2\rho_1 + (\rho_2)^2, \\ &= (\rho_1)^2 + (\rho_2)^2, \\ &= \rho_1 + \rho_2. \end{aligned} \tag{6.25}$$

The sum is not primitive so it is not an "elementary particle", but it is an idempotent so we can think of it as two elementary particles at the same point

in space.

On the other hand, if we add two identical idempotents, their sum is not an idempotent:

$$(\rho_1 + \rho_1)^2 = 2(\rho_1 + \rho_1). \tag{6.26}$$

We will interpret this as the primitive idempotent form of the Pauli exclusion principle. This book is devoted to discrete degrees of freedom in the operator formalism of quantum mechanics. The author plans to demonstrate the Pauli exclusion interpretation in a later book devoted to spatial dependencies and continuous degrees of freedom.

The set of all possible sums of distinct primitive idempotents in our set generated by ι_1, ι_2, and ι_3 includes $2^8 = 256$ elements as each of the eight primitive idempotents can either be included, or not, in the sum. Ouch.

In Sec. (5.3) we saw that velocity is a natural quantum number of a Clifford algebra. The operator for velocity in the $+z$ direction is $\widehat{zt}$. Let us suppose that this is ι_3, one of the commuting roots of unity. Now the eight primitive idempotents are split into two groups of four each. The set $\{\rho_{---}, \rho_{--+}, \rho_{-+-}, \rho_{-++}\}$ are eigenstates of $\widehat{zt}$ with eigenvalue -1 while $\{\rho_{+--}, \rho_{+-+}, \rho_{++-}, \rho_{+++}\}$ are eigenstates with eigenvalue $+1$. These two eigenvalues correspond to movement in the $-z$ and $+z$ directions, and so composite particles made from primitive idempotents selected from both these two groups cannot share the same point in space for longer than an instant.

Furthermore, because we require physics to be equivalent under rotations, the particles traveling in the $-z$ direction are identical to the particles traveling in the $+z$ direction and we need only study one of these sorts of particles to understand the others. Therefore, in this section we will restrict our attention to primitive idempotents that take a $+1$ eigenvalue with respect to $\widehat{zt}$. These are primitive idempotents that are traveling in the same direction. We will call a group of primitive idempotents that travel together like this a "snuark", and they will be needed to classify the elementary fermions. Since there are only 4 primitive idempotents with an eigenvalue for $\widehat{zt}$ of $+1$, there are only $2^4 = 16$ such snuarks and many of them are trivial.

In order to compare with the elementary particles of the standard model, we need to compile quantum numbers for these sixteen composite idempotents (snuarks). In the standard model of quantum mechanics, quantum numbers are obtained as the eigenvalues of operators. We have followed this tradition with the primitive idempotents, but if we continue the practice in the composite idempotents we will end up with nonsensical results.

The difficulty arises from the fact that the standard model is built from

spinors. If one has two distinct particle types and one has spinors for each of them that are eigenstates of an operator A with two eigenvalues, for example:

$$\begin{array}{rcl} A|a1\rangle & = & a_1|a1\rangle, \\ A|a2\rangle & = & a_2|a2\rangle, \end{array} \tag{6.27}$$

then the combined state is also an eigenstate of A, but with an eigenvalue given by the sum of the individual eigenvalues:

$$\begin{array}{rcl} A(|a1\rangle|a2\rangle) & = & (A|a1\rangle)\,|a2\rangle + |a1\rangle\,(A|a2\rangle), \\ & = & a_1(|a1\rangle|a2\rangle) + a_2(|a1\rangle|a2\rangle), \\ & = & (a_1+a_2)(|a1\rangle|a2\rangle). \end{array} \tag{6.28}$$

Therefore, to match the results for spinors, we must add the eigenvalues of the primitive idempotents that contribute to the composite idempotent.

Earlier, in the expansion of the primitive idempotents, we noted that the eigenvalue of a primitive idempotent with respect to an operator was given by the sign of the term in the primitive idempotent's expansion. When we add primitive idempotents, these values will add. Therefore, we see that our technique for assigning eigenvalues to snuarks can be defined more simply than by just copying the spinor method. The eigenvalue of a snuark is just the coefficient of the operator in the expansion of the snuark (and we multiply by 2^M where M is the number of commuting roots of unity so that the eigenvalues of a primitive idempotent are ± 1).

This concept is important and is less confusing than it may appear. Consider the snuark $\rho = \rho_{+--} + \rho_{+-+} + \rho_{+++}$. What is its quantum number for ι_1? There are two ways to compute. First,

$$\begin{array}{rcl} \iota_1\rho_{+--} & = & -\rho_{+--}, \\ \iota_1\rho_{+-+} & = & +\rho_{+-+}, \\ \iota_1\rho_{+++} & = & +\rho_{+++}, \end{array} \tag{6.29}$$

so there are two primitive idempotents with quantum number $+1$, and one with quantum number -1. Therefore the answer is $-1+1+1=+1$. But note that the snuark is *not* an eigenstate of ι_1. Instead, $\iota_1\,\rho$ gives $-\rho_{+--}+\rho_{+-+}+\rho_{+++}$.

As a second way of finding the eigenvalue of ρ with respect to ι_1, we can expand ρ_{+--}, ρ_{+-+}, and ρ_{+++} and add them together:

$$\begin{array}{rcl} \rho & = & \rho_{+--} + \rho_{+-+} + \rho_{+++}, \\ & = & 0.125(1 + \iota_3 - \iota_2 - \iota_1 + \iota_2\iota_1 - \iota_3\iota_1 - \iota_3\iota_2 + \iota_3\iota_2\iota_1) \\ & & +0.125(1 + \iota_3 - \iota_2 + \iota_1 - \iota_2\iota_1 + \iota_3\iota_1 - \iota_3\iota_2 - \iota_3\iota_2\iota_1) \\ & & +0.125(1 + \iota_3 + \iota_2 + \iota_1 + \iota_2\iota_1 + \iota_3\iota_1 + \iota_3\iota_2 + \iota_3\iota_2\iota_1) \\ & = & 0.125(3 + 3\iota_3 - \iota_2 + \iota_1 + \iota_2\iota_1 + \iota_3\iota_1 - \iota_3\iota_2 + \iota_3\iota_2\iota_1). \end{array} \tag{6.30}$$

Since the coefficient of ι_1 in the sum is $+1$, (after multiplying by $2^3 = 8$ to get rid of the common factor of 0.125) this is the eigenvalue. The other

eigenvalues can be read off the last line. For example, the eigenvalue of ι_3 is $+3$, and the eigenvalue of $\iota_3\iota_1$ is $+1$. Note that while the quantum numbers of a primitive idempotent are multiplicative, the quantum numbers of a snuark are not. Instead, since ι_3 has the same eigenvalue $+1$, for these four primitive idempotents, the eigenvalue of ι_1 is the same as the eigenvalue for $\iota_3\iota_1$.

In combining the 4 primitive idempotents ρ_{+--}, ρ_{+-+}, ρ_{++-} and ρ_{+++} into sixteen possible snuarks, only some of them are of interest to us. The snuarks follow the binomial theorem in that there is 1 with no primitive idempotents, 4 with exactly one, 6 with exactly two, 4 with three, and 1 with all four. The one with no primitive idempotent is simply zero and has no interpretation. The four with single primitive idempotents are simply the primitive idempotents themselves and we already know these well. When all four primitive idempotents are added together the result is $\hat{1}$ which will take eigenvalues of $+1$ with respect to all operators, and so is not of interest. The four with exactly three primitive idempotents will carry eigenvalues equal to the negative of the eigenvalues of the missing primitive idempotent. This could be equally accomplished by using primitive idempotents and negating the operators so these four are also not of interest.

The only snuarks of interest are the ones that have two primitive idempotents. Their eigenvalues are as follows:

	ρ_{+++}	ρ_{++-}	ρ_{+-+}	ρ_{+--}	ι_2	ι_1	ι_{21}
ρ_{1100}	1	1			$+2$	0	0
ρ_{1010}	1		1		0	$+2$	0
ρ_{1001}	1			1	0	0	-2
ρ_{0011}			1	1	-2	0	0
ρ_{0101}		1		1	0	-2	0
ρ_{0110}		1	1		0	0	$+2$

(6.31)

In the above, we use, ρ_{abcd} to designate which of the four primitive idempotents, $(\rho_{+++}, \rho_{++-}, \rho_{+-+}, \rho_{+--})$ are included in the snuark. In a later chapter we will use these to model the elementary fermions.

6.6 Generalized Stern-Gerlach Plots

The original Stern-Gerlach experiment split a beam of atoms in two. This we modeled with the two primitive idempotents built from a single root of unity. More complicated Clifford algebras allow larger numbers of commuting roots of unity which define larger numbers of primitive idempotents. In this section we discuss generalized Stern-Gerlach experiments with these more general particle beams.

With two commuting roots of unity, we have four primitive idempotents. There are then four primitive Stern-Gerlach filters, and the associated Stern-

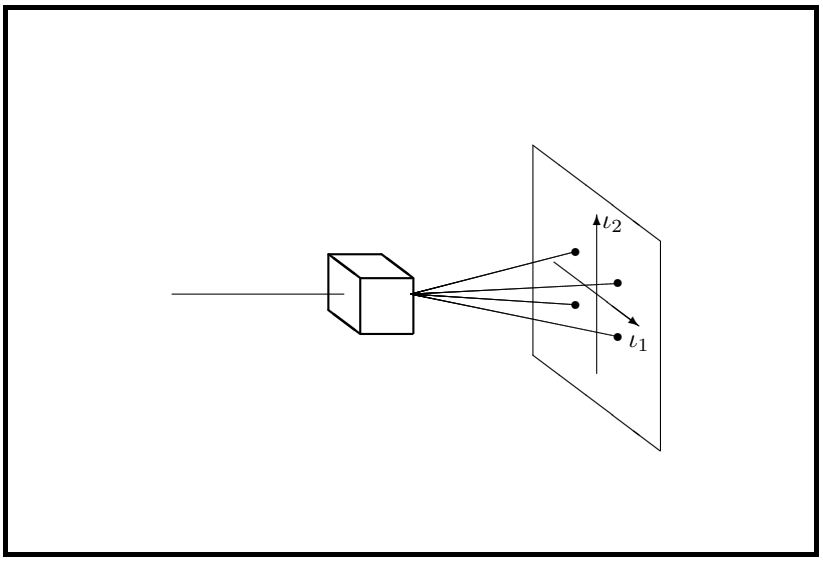


Figure 6.3: Stern-Gerlach experiment that splits a beam of particles according to two commuting quantum numbers, ι_1 and ι_2.

Gerlach experiment will have four outgoing beams instead of just two. See Fig. (6.3). The choice of the operators ι_1 and ι_2 is arbitrary. We could have chosen $-\iota_2$ and $\iota_1\iota_2$ and the diagram would look exactly the same. This is because the eigenvalues of the four primitive idempotents are symmetric with respect to negation and swapping any two operators:

	ι_2	ι_1	$\iota_2\iota_1$
ρ_{+--}	$-$	$-$	$+$
ρ_{+-+}	$-$	$+$	$-$
ρ_{++-}	$+$	$-$	$-$
ρ_{+++}	$+$	$+$	$+$

(6.32)

That is, no matter which 2 of the operators are chosen out of $\{\pm\iota_2, \pm\iota_1, \pm\iota_2\iota_1\}$, there will be one primitive idempotent which has eigenvalues of $(-1,-1)$, one which has eigenvalues of $(+1,+1)$, and two with mixed eigenvalues of $(-1,+1)$ and $(+1,-1)$. No matter which operators are chosen for the Stern-Gerlach apparatus, the result is the same, a square.

In these Stern-Gerlach experiments, the screen is 2-dimensional, so there is only room for two operators to be explored on it. This will not prevent our imagination from exploring Stern-Gerlach experiments with three operators. Instead of defining the beam splitting by sending the beams to a screen, we will instead draw the beams as spots in 3-dimensions, with the eigenvalues giving the coordinates of each spot. If we have three eigenvalues that are all ± 1, the spots will fall on the corners of a cube.

As our first 3-dimensional Stern-Gerlach experiment, we will draw one with the three non independent operators ι_1, ι_2, and $\iota_2\iota_1$. Of course these define only four states, so we will only have four spots instead of the eight that one might expect from three operators. The resulting 3-dimensional Stern-Gerlach drawing is shown as Fig. (6.4)

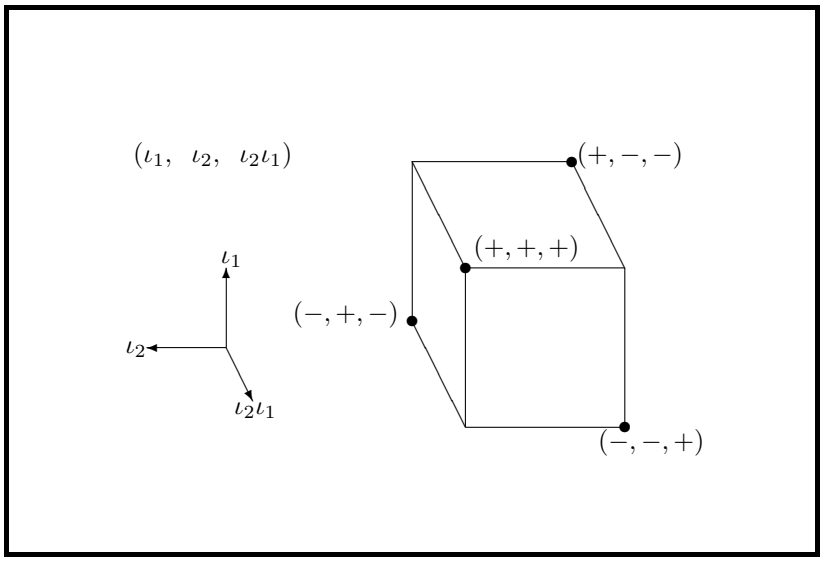


Figure 6.4: Stern-Gerlach experiment with the three non independent operators, ι_1, ι_2, and $\iota_2\iota_1$. Spots are labeled with their eigenvalues, ± 1, in the order $(\iota_1, \iota_2, \iota_2\iota_1)$. The cube is drawn only for help in visualization.

When we assume that the particle beam is traveling in the same direction we end up eliminating one of the commuting roots of unity from use in describing the particle type of the beam. This halves the number of primitive idempotents that we have available for modeling particles. With 3 commuting roots of unity, we get only $2^2 = 4$ different particles. But if these are all particles, we can suppose that corresponding to each there is an antiparticle that can be modeled by the particle traveling backwards in time.

So long as we distinguish between particles and antiparticles, this gives us back our $2^3 = 8$ different quantum states. In this case, we can use ι_1, ι_2, and ι_3 as independent operators and our quantum states will occupy the corners of a cube. We can think of ι_3 as an operator that distinguishes between particles (with eigenvalues of $+1$) and antiparticles (with eigenvalues of -1). If we choose ι_3 as one of the three operators that define the Stern-Gerlach beam plot, we will have that the particles make up the four corners of the cube surrounding

one side, and the other four corners make up the antiparticles.

It is important to realize that the distribution of the primitive idempotent quantum states on the corners of the cube is very arbitrary in that it depends entirely on our choice of three independent operators. Rather than choosing ι_1, ι_2, and ι_3, we can choose any other set of independent operators and we can choose them in any order. For example, $\iota_1\iota_2$, $\iota_2\iota_3$ and $-\iota_2$ are three independent commuting roots of unity (i.e. operators) and will define a beam pattern for a Stern-Gerlach experiment. See Fig. (6.5).

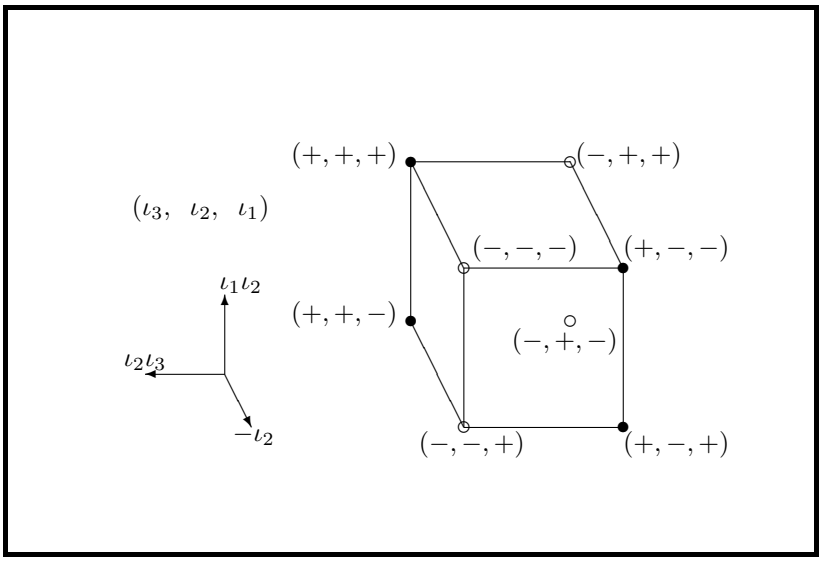


Figure 6.5: Stern-Gerlach plot with the three operators $(\iota_1\iota_2,\ \iota_2\iota_3,\ -\iota_2)$. The labels for each corner are the ι_1, ι_2 and ι_3 quantum numbers in order $(\iota_3,\ \iota_2,\ \iota_1)$. The antiparticles have $\iota_3 = -1$ and are drawn as hollow circles.

As a final example of generalized Stern-Gerlach plots, we will plot the six snuarks whose eigenvalues are shown in Eq. (6.31). Since the eigenvalues of ι_1 and ι_2 are 0 or ± 2, we will divide these eigenvalues by 2 so that they fit more elegantly in cube form.

It is traditional to have antiparticles carry negative quantum numbers. To accomplish this, rather than plot the ι_1 and ι_2 quantum numbers, we will instead plot $\iota_3\iota_1$ and $\iota_3\iota_2$. Since ι_3 will be -1 for the antiparticles, this will multiply the antiparticle quantum numbers by -1. With the antiparticles, we will have a total of 12 quantum states to plot. The quantum numbers for the

six composite particles are:

	ρ_{+++}	ρ_{++-}	ρ_{+-+}	ρ_{+--}	ι_3	$\iota_3\iota_2/2$	$\iota_3\iota_1/2$
ρ_{1100}	1	1			$+1$	$+1$	0
ρ_{1010}	1		1		$+1$	0	$+1$
ρ_{1001}	1			1	$+1$	0	0
ρ_{0011}			1	1	$+1$	-1	0
ρ_{0101}		1		1	$+1$	0	-1
ρ_{0110}		1	1		$+1$	0	0

(6.33)

and the quantum numbers for the antiparticles are the negatives of the above. Note that two of the snuarks, ρ_{1001} and ρ_{0110} appear degenerate, but if we also plotted $\iota_2\iota_1$ (perhaps by adding a 4th dimension to the plot) the degeneracy would be broken. The resulting 12 particles in 11 spots are shown in Fig. (6.6). The apparently degenerate particles will be eliminated by energy considerations in the chapter on Force.

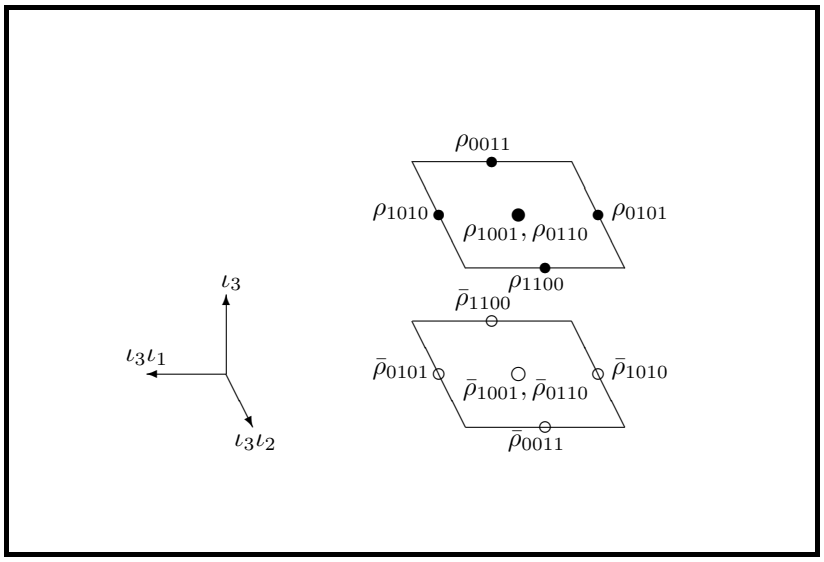


Figure 6.6: Stern-Gerlach plot of 12 snuarks and their antiparticles. See Eq. (6.33) for the quantum numbers. The antiparticles have $\iota_3 = -1$ and are drawn as hollow circles.

Chapter 7

Force

Yet sleek with oil, a Force was hid
    Making mock of all they did
Ready at the opening hour
To yield up to Prometheus
The secular and well-drilled Power
The Gods secreted thus.

We need to define a force that will bind together primitive idempotents in a natural way. Since the elementary fermions are point-like particles, the force will have to create bound states that act point-like. To arrange this, we will define the force as a potential energy that defines the total energy of a set of primitive idempotents that are at the same point in space.

7.1 Potential Energy, Some Guesses

The presence of three generations of elementary particles naturally leads one to suppose that the higher generations are some sort of excitation of the first generation. This would explain the hierarchy between generations. However, experiments measuring the total cross section for $e + \bar{e} \rightarrow$ hadrons show that if there are four generations of elementary particles, the mass of the 4th neutrino has to be on the order of 10 GeV/c^2, which is unnaturally heavy.[19, sec 11.2] This result argues that there should only be three generations, no more and no less.

A natural way of obtaining exactly three generations of relatively low mass particles is to suppose that, to first order, they are a degenerate ground state and the masses we measure just come from a second order splitting of the degeneracy. This is the assumption we will make.

Assuming that the elementary fermions are degenerate removes some of the difficulty of assigning a potential energy as we need only correctly guess the potential energy in the neighborhood of the degeneracy. There are several different forms for the potential energy that come to mind and we will discuss them in this section.

In analyzing the Stern-Gerlach experiments of the previous chapter, we saw that a primitive idempotent represented three things at once. It represents an elementary particle. It projects out the portion of the incoming beam that consists of the elementary particle. And it represents the field configuration in the Stern-Gerlach filter that separates that elementary particle from all others. We can treat this last observation as a clue to the form of the potential energy.

Suppose that our universe of particles has only two particles in it, a positive charged one and a negatively charged one, where "charge" we leave nebulously defined. Only one Stern-Gerlach experiment exists, one that separates positively charged particles from negatively charged ones. The two particles are represented by two primitive idempotents:

$$\begin{array}{rcl} \rho_+ & = & 0.5(1+\hat{c}), \\ \rho_- & = & 0.5(1-\hat{c}), \end{array} \tag{7.1}$$

where $\hat{c}$ is the root of unity, or operator, defining the charge.

There is only one bound state, that consisting of a positively charged particle and a negatively charged particle. We need a potential energy that will bind two oppositely charged particles together. As a function, potential energy needs to be real valued and to give zero only when the two states are opposite in charge.

In the spinor language, a natural choice for potential energy that meets these requirements is the probability of transition between the states. That is, since the two primitive idempotents annihilate, there will be a zero probability of transition between them. We will write this guess as V_1:

$$V_1(A,B) = \langle A|B\rangle\langle B|A\rangle = |\langle A|B\rangle|^2. \tag{7.2}$$

n $|A\rangle$ is multiplied by an arbitrary phase, $\langle A|$ is multiplied by its complex conjugate so the above does not depend on the arbitrary spinor phase. We can rewrite this in operator form by using the trace function:

$$\begin{array}{rcl} V_1(\rho_A,\rho_B) & = & \mathrm{tr}(\rho_A\ \rho_B). \\ & = & \mathrm{tr}(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}), \\ & = & a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}. \end{array} \tag{7.3}$$

The above potential will work well for a more complicated algebra than one that just allows two oppositely charged particles. We can assume that $|A\rangle$ and $|B\rangle$ are in the Pauli algebra. In the usual quantum mechanics, the Pauli exclusion principle would prevent A and B from occupying other than opposite spin states. Here we instead allow them to occupy arbitrary states and obtain the Pauli exclusion principle as a result of presuming that the known particles are all ground states of this potential energy.

That the energy increases when A and B are not in opposite spin states fits in well with our intuition. When A and B are distinct particles the Pauli exclusion principle does not prevent them from occupying the same spin state and in such cases, one typically finds that the energy is slightly lower when the two particles are oppositely arranged. An example of this effect is atomic hydrogen.

To simplify Eq. (7.3), of course we put it into geometric form. To do this, recall that one converts a 2×2 complex matrix into geometric form as follows:

$$\begin{aligned} a_1 &= (a_{11} + a_{22})/2, \\ a_x &= (a_{12} + a_{21})/2, \\ a_y &= i(a_{12} - a_{21})/2, \\ a_z &= (a_{11} - a_{22})/2. \end{aligned} \tag{7.4}$$

The inverse relations are then

$$\begin{aligned} a_{11} &= a_1 + a_z, \\ a_{12} &= a_x + i a_y, \\ a_{21} &= a_x - i a_y, \\ a_{22} &= a_1 - a_z. \end{aligned} \tag{7.5}$$

Note that in order for our density matrices ρ_A and ρ_B must be Hermitian, and therefore a_1, a_x, a_y, and a_z are all real. Substituting these relations into Eq. (7.3) we obtain:

$$\begin{aligned} V(A,B) &= (a_1 + a_z)(b_1 + b_z) + (a_x + i a_y)(b_x + i b_y) \\ &\quad + (a_x - i a_y)(b_x - i b_y) + (a_1 - a_z)(b_1 - b_z), \\ &= 2(a_1 b_1 + a_x b_x + a_y b_y + a_z b_z). \end{aligned} \tag{7.6}$$

This is exactly what we would obtain if we treated the degrees of freedom of ρ_A and ρ_B as if they were real, Euclidean 4-vectors and defined the potential energy as the dot product. Thus we see that a potential energy that we defined as a 4th order function in spinors becomes a tractable 2nd order function in geometric density operators.

In choosing the potential energy to be the transition probability between the states we found a reasonable potential energy for two states, but what we need is a potential energy that can bind together an arbitrary number of primitive idempotents. To get this sort of object, we can consider summing together the states to be bound, and define the potential energy as a function of the sum.

As a first step at putting the potential energy in the form of a summation of primitive idempotents, let us note that $\rho_+ + \rho_- = 1$. Therefore we might suppose that bound states of primitive idempotents must always sum to unity.

Then we can define the potential as being the square of the difference between the sum and unity.

$$V_2(\rho_A, \rho_B) = \left|\rho_A + \rho_B - \hat{1}\right|^2 . \tag{7.7}$$

To define $|\ \ |^2$, we simply take the sum of squared magnitudes of the elements of the matrix. This gives:

$$\begin{aligned} V_2(\rho_A, \rho_B) &= \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right|^2 , \\ &= |a_{11} + b_{11} - 1|^2 + |a_{12} + b_{12}|^2 \\ &\quad + |a_{21} + b_{21}|^2 + |a_{22} + b_{22} - 1|^2. \end{aligned} \tag{7.8}$$

To simplify the above, we again use Eq. (7.5) to put it into geometric form. Remembering that a_1, a_x, ... b_z are all real we have:

$$\begin{aligned} V_2(\rho_A, \rho_B) &= |a_1 + a_z + b_1 + b_z - 1|^2 + |a_x + ia_y + b_x + ib_y|^2 \\ &\quad + |a_x - ia_y + b_x - ib_y|^2 + |a_1 - a_z + b_1 - b_z - 1|^2, \\ &= (a_1 + b_1 + a_z + b_z - 1)^2 + 2(a_x + b_x)^2 + 2(a_y + b_y)^2 \\ &\quad + (a_1 + b_1 - a_z - b_z - 1)^2, \\ &= 2(a_1 + b_1)^2 + 2(a_x + b_x)^2 + 2(a_y + b_y)^2 + 2(a_z + b_z)^2 \\ &\quad - 4(a_1 + b_1) + 2, \\ &= 2(a_1^2 + a_x^2 + a_y^2 + a_z^2) + 2(b_1^2 + b_x^2 + b_y^2 + b_z^2) \\ &\quad + 4(a_1 b_1 + a_x b_x + a_y b_y + a_z b_z) - 4(a_1 + b_1) + 2. \end{aligned} \tag{7.9}$$

For a normalized primitive idempotent, $a_1 = 0.5$, and $a_x^2 + a_y^2 + a_z^2 = 0.25$, so the above can be considerably simplified:

$$\begin{aligned} V_2(\rho_A, \rho_B) &= 2(0.25 + 0.25) + 2(0.25 + 0.25) \\ &\quad + 4(a_1 b_1 + a_x b_x + a_y b_y + a_z b_z) - 4(0.5 + 0.5) + 2, \\ &= 4(a_1 b_1 + a_x b_x + a_y b_y + a_z b_z). \end{aligned} \tag{7.10}$$

Comparing with Eq. (7.6), we see that $V_2(A, B) = 2V_1(A, B)$. In other words, these two very different approaches to the potential energy gave identical potential energies, except for a scaling factor.

Another way of defining the potential energy is to suppose that there is some sort of statistical mechanical basis underlying quantum mechanics, and use the entropy function defined in Eq. (6.8). We suppose that the potential energy is minimized when the entropy is maximized so we need a minus sign to get a guess at an energy function:

$$V_3(\rho_A, \rho_B) = k\ \ \mathrm{tr}((\rho_A + \rho_B)\ \ln(\rho_A + \rho_B)) \quad \text{1st guess.} \tag{7.11}$$

The entropy is written under the assumption that ρ is normalized in the usual quantum mechanical way, that is, that $\mathrm{tr}(\rho) = 1$. For the Pauli algebra,

this means that $\rho = 0.5\hat{1}$. Following the hint provided by the Stern-Gerlach operators, we need a potential energy $V_3(\rho_A + \rho_B)$ that is minimized when $\rho_A + \rho_B = \hat{1}$.

In addition, in terms of geometry, the trace, when applied to the matrix representation of a Clifford algebra element, picks out the scalar part. Therefore, to convert the entropy formula into a potential energy we must write:

$$V_3(\rho_A, \rho_B) = V_3(\rho_A + \rho_B) = k\ \langle 0.5(\rho_A + \rho_B) \ln(0.5(\rho_A + \rho_B))\rangle_0, \qquad (7.12)$$

where $\langle \quad \rangle_0$ is the notation for the picking out of the scalar part. Since the entropy is maximized when $\rho = 0.5\hat{1}$, the above will be minimized when $\rho_A + \rho_B = \hat{1}$.

Since we are assuming that the potential energy is of the order of the Planck mass, we need only worry about small perturbations around the minimum. As before, $\rho_A = 0.5\hat{1} + a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ and similarly for ρ_B. "Small perturbations" means that $a_x + b_x$ is very small, and similarly for y and z so:

$$\rho_A + \rho_B = \hat{1} + (a_x + b_x)\hat{x} + (a_y + b_y)\hat{y} + (a_z + b_z)\hat{z}, \qquad (7.13)$$

where all the coefficients but the first are very small. We write: $\rho_A + \rho_B = 1 + \delta$. Putting this into Eq. (7.12) and keeping up to second order terms gives:

$$\begin{aligned} V_3(\rho_A, \rho_B) &= k\ \langle (\hat{1} + 0.5\delta) \ln(0.5(\hat{1} + 0.5\delta))\rangle_0, \\ &= k\ \langle (1 + 0.5\delta)(\ln(0.5) + \ln(1 + 0.5\delta))\rangle_0, \\ &= k\ \langle (\ln(0.5) + \ln(1 + 0.5\delta)) + 0.5\delta(\ln(0.5) + \ln(1 + 0.5\delta))\rangle_0, \\ &= k\ \langle (\ln(0.5) + 0.5\delta - 0.5\delta^2 + 0.5\delta \ln(0.5) + 0.25\delta^2)\rangle_0, \\ &= k\ \langle (\ln(0.5) + 0.5\delta(1 + \ln(0.5)) - 0.25\delta^2). \end{aligned} \qquad (7.14)$$

Note that the $\langle \quad \rangle_0$ keeps only the scalar part and δ is purely vector. Therefore the above reduces to:

$$\begin{aligned} V_3(\rho_A, \rho_B) &= k\ (\ln(0.5) - 0.25\delta^2), \\ &= k\ (\ln(0.5) - 0.25((a_x + b_x)^2 + (a_y + b_y)^2 + (a_z + b_z)^2)), \\ &= k\ (\ln(0.5) - 0.25(a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2) \\ &\quad +0.5(a_x b_x + a_y b_y + a_z b_z)), \\ &= k\ (\ln(0.5) - 0.125 - 0.5(a_x b_x + a_y b_y + a_z b_z)), \\ &= k\ (\ln(0.5) - 0.5(a_1 b_1 + a_x b_x + a_y b_y + a_z b_z)). \end{aligned} \qquad (7.15)$$

The $0.5 \ln(0.5)$ is just an additive constant. The rest of the potential energy is simply a multiple of the same potential energy we obtained for V_1 and V_2.

7.2 Geometric Potential Energy

The transition probability V_1 is difficult to generalize to n primitive idempotents. Entropy, V_3 has the disadvantage of having a logarithm. And V_2 is written for two particles only. But with these as clues, we can make a fourth

guess:

$$V_4(\rho_A, \rho_B, ...\rho_\chi) = (a_x + b_x + ... + \chi_x)^2 + (a_y + b_y + ... + \chi_y)^2 + (a_z + b_z + ... + \chi_z)^2. \tag{7.16}$$

That is, we sum the primitive idempotents, and calculate a squared magnitude that includes everything but the scalar part. When the primitive idempotents sum to a scalar, this gives a potential energy of zero which is the minimum.

For a Clifford algebra larger than the Pauli algebra, we can extend the sum on the right to include the rest of the non scalar degrees of freedom. But in the above we are assigning different relative weights to the scalar and vector components, should we also assign differing weights to the bivector components?

If we were not assuming that the elementary particles are states with zero energy, the natural interpretation of their different energies would be that they give their respective masses. So to break the degeneracy, we can rewrite V_4 so that the scalar part of the sum does contribute to the potential energy. This will still keep the particles as near degenerate in energy, but will allow us to model the masses of the elementary particles.

Writing the potential energy with different weights for the scalar and non scalar parts raises the question of how we should assign weights for all the various non scalar parts. The most complicated Clifford algebra we are considering is $C(4,1)$, the complex algebra generated by $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{s}$ and $\hat{t}$ where only the last has negative signature. This Clifford algebra has 32 complex degrees of freedom or 64 real degrees of freedom.

Assuming that we give the potential energy full freedom to vary over the 32 complex degrees of freedom, we can write V using 32 possibly different weights, v_χ:

$$\begin{aligned} & V(a_1\hat{1} + a_x\hat{x} + a_y\hat{y} + ... + a_{xyzst}\widehat{xyzst}), \\ = \; & V(a_1\hat{1}) + V(a_x\hat{x}) + V(a_y\hat{y}) + ... + V(a_{xyzst}\widehat{xyzst}), \\ = \; & v_1|a_1|^2 + v_x|a_x|^2 + v_y|a_y|^2 + ... + v_{xyzst}|a_{xyzst}|^2. \end{aligned} \tag{7.17}$$

We need to choose the 32 real values, v_χ, a task which appears formidable.

The various degrees of freedom for a Clifford algebra are related by multiplication. For example, $\widehat{xy} = \hat{x}\hat{y}$. It is therefore natural to write the potential energy function as a multiplicative function. For example:

$$v_{xyzst} = v_x \; v_y \; v_z \; v_s \; v_t. \tag{7.18}$$

This means that instead of defining 32 real values, we need only define 5, and of those, it is natural to choose the spatial potentials as:

$$v_x = v_y = v_z = v_s. \tag{7.19}$$

The scalar potential for $\hat{1}$ has to be the multiplicative identity:

$$V(\hat{1}) = v_1 = 1. \tag{7.20}$$

Then we can write the potential V completely in terms of v_s and v_t. An example potential energy calculation:

$$\begin{array}{rcl} V(\rho) & = & V(0.25(\hat{1} + \widehat{zt})(\hat{1} - \hat{s})), \\ & = & V(0.25(\hat{1} + \widehat{zt} - \hat{s} + \widehat{zst})), \\ & = & |0.25|^2 + v_s v_t |0.25|^2 + v_s |0.25|^2 + v_s^2 v_t |0.25|^2, \\ & = & 0.0625(1 + v_s v_t)(1 + v_s). \end{array} \tag{7.21}$$

Thus we can write potential energies as polynomials in v_s and v_t that factor the same way as primitive idempotents factor into idempotents.

The leptons are observed as free particles but the primitive idempotents described in this book do not. In order for this to be a result of energy considerations, we need to have the energies of a typical scalar be much smaller than the energy of a typical primitive idempotent. For example,

$$\begin{array}{rcl} V(\hat{1}) & << & V(0.25(\hat{1} + \widehat{zt})(\hat{1} - \hat{s})), \quad \text{or} \\ 1 & << & 0.625(1 + v_s v_t)(1 + v_s). \end{array} \tag{7.22}$$

The easy way to achieve this is to assume that v_s and v_t are much greater than 1, but we do not need to assume that these are the same.

The potential energy function defines a map from any element of the Clifford algebra to the non negative reals. In particular, it is multiplicative on a substantial number of the commuting roots of unity. For example:

$$\begin{array}{rcl} V(\widehat{ixyzt}) & = & v_x v_y v_z v_t, \\ & = & V(\widehat{ixy})\ V(\widehat{zt}). \end{array} \tag{7.23}$$

As we saw in the chapter on Primitive Idempotents, a set of commuting roots of unity form an Abelian group. Without having defined the potential, we had no way of preferring one way of generating the group from another. For example, one could write an idempotent in either of these equivalent ways:

$$\begin{array}{r} 0.25(1 + \widehat{ixy})(1 + \widehat{zt}), \\ 0.25(1 + \widehat{ixy})(1 + \widehat{ixyzt}), \end{array} \tag{7.24}$$

The equivalence boils down to the fact that the Abelian group of commuting roots of unity, $\{1, \widehat{ixy}, \widehat{zt}, \widehat{ixyzt}\}$ can be generated by either $\{\widehat{ixy}, \widehat{zt}\}$ or $\{\widehat{ixy}, \widehat{ixyzt}\}$.

Let us work on understanding the potential energy structure of the commuting roots of unity of the most complicated Clifford algebra we have considered, $C(4,1)$. Since $\widehat{ixyzst}$ squares to $+1$ and commutes with everything, it must be included. It remains to choose two independent roots of unity. Here, "independent" means with respect to $\widehat{ixyzst}$. For example, $\widehat{ixy}$ and $\widehat{zst}$ are not independent since they multiply to give $\widehat{ixyzst}$. Because multiplication by $\widehat{ixyzst}$ will have the effect of changing the number of vectors in a root of unity from q to $5-q$, without loss of generality we can suppose that the two roots of unity we add to $\widehat{ixyzst}$ will have 2 or fewer vectors.

Since two vectors anticommute, we cannot add two roots of unity that are both made from one vector such as $\hat{x}$ or $\hat{y}$. And if the two roots we add both have two vectors, for example $\widehat{ixy}$ and $\widehat{zt}$, then the only way they can commute is if they share no vectors. In that case, their product will have four distinct vectors, and the product of that with $\widehat{ixyzst}$ will have only one vector, in this case $\hat{s}$. So we need only consider the cases where we add a vector and a bivector. For these to commute, they can share no vectors.

Keeping track only of the only time vector, $\hat{t}$ and the four space vectors in no particular order, $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$, the only possible choices for a complete set of independent commuting roots of unity are the following:

$$\begin{array}{c} \{\widehat{it},\ \widehat{iab},\ \widehat{icd}\}, \\ \{\hat{a},\ \widehat{ibc},\ \widehat{dt}\}. \end{array} \tag{7.25}$$

That is, either $\widehat{it}$ is used as a root of unity, or it is in one of the generating roots. Since we will be wanting our primitive idempotents to be eigenstates of velocity, such as $\widehat{zt}$, we take the second case.

Writing out the full set of 8 commuting roots of unity, we can read off the potential energies of the operators:

$\hat{1}$	$\widehat{iab}$	$\widehat{icd}$	$\widehat{abcd}$	$\widehat{it}$	$\widehat{iabt}$	$\widehat{icdt}$	$\widehat{ixyzst}$
1	v_s^2	v_s^2	v_s^4	v_t	$v_t v_s^2$	$v_t v_s^2$	$v_t v_s^4$

(7.26)

If $v_s < v_t$, then the smallest potential energy is $v_t v_s$. Otherwise, it is, v_s^2. But in any case, the largest contribution has to be from the $\widehat{ixyzst}$ term.

7.3 Snuarks as Bound States

The fact that $\widehat{ixyzst}$ is the largest contribution to the energy of a primitive idempotent suggests that they will be deeply bound together in order to cancel this contribution. In order to describe the ways that this can happen, let's specialize to primitive idempotents traveling in the $+z$ direction.

In order to travel in the $+z$ direction, the primitive idempotents must carry a factor of $(\hat{1} + \widehat{zt})$ as was described in Sec. (5.3). In order for their sum to

cancel the $i\widehat{xyzst}$ term, we can add two primitive idempotents, one with a factor of $(\hat{1} + i\widehat{xyzst})$, the other with a factor of $(\hat{1} - i\widehat{xyzst})$. There remains one primitive idempotent to specify, it is generated by a commuting root of unity that must commute with $\widehat{zt}$. Of course anything commutes with $i\widehat{xyzst}$.

The stuff that commutes with $\widehat{zt}$ is generated by the four elements $\{\hat{s}, \hat{x}, \hat{y}, \widehat{zt}\}$. Somewhat arbitrarily, we will assume that the third commuting root of unity is $\hat{s}$. We could, for example, have chosen $i\widehat{xyzt}$ or $\widehat{ixy}$, but these have higher potential energy than $\hat{s}$.

We can suppose that the primitive idempotents bound by the cancelation of $i\widehat{xyzst}$ can have either eigenvalue with respect to $\hat{s}$. The four primitive idempotents that we will consider the binding potential on are therefore:

$$\begin{aligned}
\rho_{z--} &= 0.125(1 + \widehat{zt})(1 - \hat{s})(1 - i\widehat{xyzst}), \\
\rho_{z-+} &= 0.125(1 + \widehat{zt})(1 - \hat{s})(1 + i\widehat{xyzst}), \\
\rho_{z+-} &= 0.125(1 + \widehat{zt})(1 + \hat{s})(1 - i\widehat{xyzst}), \\
\rho_{z++} &= 0.125(1 + \widehat{zt})(1 + \hat{s})(1 + i\widehat{xyzst}),
\end{aligned} \tag{7.27}$$

Multiplying these out we have:

$$\begin{aligned}
\rho_{z--} &= 0.125(1 + \widehat{zt} - \hat{s} - i\widehat{xyzst} + \widehat{zst} + i\widehat{xys} - i\widehat{xyzt} - \widehat{ixy}), \\
\rho_{z-+} &= 0.125(1 + \widehat{zt} - \hat{s} + i\widehat{xyzst} + \widehat{zst} - i\widehat{xys} + i\widehat{xyzt} + \widehat{ixy}), \\
\rho_{z+-} &= 0.125(1 + \widehat{zt} + \hat{s} - i\widehat{xyzst} - \widehat{zst} + i\widehat{xys} + i\widehat{xyzt} + \widehat{ixy}), \\
\rho_{z++} &= 0.125(1 + \widehat{zt} + \hat{s} + i\widehat{xyzst} - \widehat{zst} - i\widehat{xys} - i\widehat{xyzt} - \widehat{ixy}),
\end{aligned} \tag{7.28}$$

The potential energy of the above four primitive idempotents are identical, $(1+v_t v_s)(1+v_s)(1+v_s^2)/64$. For large v_s and v_t, this will be dominated by the $v_t v_s^4$ term. There are four combinations whose sums cancel the $i\widehat{xyzst}$ term: $\rho_{z--} + \rho_{z-+}$, $\rho_{z--} + \rho_{z++}$, $\rho_{z+-} + \rho_{z-+}$, $\rho_{z+-} + \rho_{z++}$. These are the snuarks. There are four of them for primitive idempotents traveling in the $+z$ direction.

Because there are only three commuting roots of unity, these four snuarks are all that we can define that cancel $i\widehat{xyzst}$. We will now continue with an analysis of the quantum numbers of the snuarks. To do this, we first explicitly write out the snuarks:

$$\begin{aligned}
\rho_{z--} + \rho_{z-+} &= 0.25(1 + \widehat{zt} - \hat{s} + \widehat{zst}), \\
\rho_{z+-} + \rho_{z++} &= 0.25(1 + \widehat{zt} + \hat{s} - \widehat{zst}), \\
\rho_{z--} + \rho_{z++} &= 0.25(1 + \widehat{zt} - i\widehat{xyzt} - \widehat{ixy}), \\
\rho_{z+-} + \rho_{z-+} &= 0.25(1 + \widehat{zt} + i\widehat{xyzt} + \widehat{ixy}).
\end{aligned} \tag{7.29}$$

The four snuarks naturally group into two pairs, according to which degrees of freedom they have not canceled out.

All four snuarks have a $\widehat{zt}$ term and are therefore eigenstates of velocity. The top two have a $\pm\hat{s}$ term and are therefore eigenstates of $\hat{s}$ with eigenvalues of ± 1. But the bottom two have no $\hat{s}$ terms; but since they are made from primitive idempotents that are eigenstates of $\hat{s}$, as composite states, they are still eigenstates of $\hat{s}$ but have eigenvalues of zero.

The primitive idempotents of the snuarks were made from three commuting roots of unity and therefore have eight quantum numbers. These are inherited by the snuarks as follows:

	1	$\widehat{zt}$	$\widehat{ixys}$	$\widehat{ixyzst}$	$\hat{s}$	$\widehat{zst}$	$\widehat{ixyzt}$	$\widehat{ixy}$
$\rho_{z--}+\rho_{z-+}$	2	2	0	0	-2	$+2$	0	0
$\rho_{z+-}+\rho_{z++}$	2	2	0	0	$+2$	-2	0	0
$\rho_{z--}+\rho_{z++}$	2	2	0	0	0	0	-2	-2
$\rho_{z+-}+\rho_{z-+}$	2	2	0	0	0	0	$+2$	$+2$

(7.30)

The first four columns have identical, and therefore uninteresting quantum numbers. The remaining four columns are each has the same structure. Two of the snuarks take zero quantum numbers, the other two take ± 2. Later we will interpret this pattern as defining the structure of the leptons, that is, a doublet and two singlets. The four snuarks will be interpreted, after a rotation by the Weinberg angle, as the left and right handed electron and neutrino.

Note that if instead of using $C(4,1)$ we were using the Dirac algebra, we would have exhausted our commuting roots of unity at $\widehat{zt}$ and $\widehat{ixyzt}$. Instead of having a list of four primitive idempotents as in Eq. (7.28), we would have a list of only two, and these two would combine in only one way. Thus instead of having four snuarks, we would have only one. This is not enough snuarks to produce the standard model fermions. In particular, it would not reproduce the structure of doublet and two singlets that characterizes weak isospin.

The snuarks defined above are composed of 2 primitive idempotents each and eliminate the $\widehat{ixyzst}$ degree of freedom. One can eliminate yet more higher degrees of freedom by adding together all four primitive idempotents:

$$\begin{aligned} \rho_{zDM} &= \rho_{z--}+\rho_{z-+}+\rho_{z+-}+\rho_{z++}, \\ &= 0.5(1+\widehat{zt}). \end{aligned} \tag{7.31}$$

This eliminates everything but the $\widehat{zt}$ non scalar degree of freedom. We will associate this snuark with dark matter, hence the abbreviation ρ_{DM}.

The electrons and neutrinos couple to the photon, $W^{\pm}$ and Z^0. We will associate these interactions with the degrees of freedom and quantum numbers of the four regular matter snuarks shown in Eq. (7.29) and Eq. (7.30). These degrees of freedom are absent in ρ_{DM}, and the quantum numbers of ρ_{DM} for these operators are zero. Therefore matter built from ρ_{DM} will not participate in electroweak interactions. This is compatible with the characteristics of dark matter postulated by the astronomers.

This book is primarily about the standard model and therefore standard matter. We will not discuss dark matter and ρ_{DM} further in this book. The topic is attractive as a research area. For example, one might suppose that big bang produced large numbers of primitive idempotents and that these primitive idempotents united together to produce standard snuarks and dark matter snuarks.

Now it is known that standard matter does not decay at an appreciable rate into dark matter. Therefore one might suppose that after the standard snuarks have condensed into standard matter, the ratio of standard snuark to dark matter snuark will be frozen, and therefore the present ratio of standard matter to dark matter will be determined by statistical mechanics and the following reversible reactions:

$$\begin{array}{lll} (\rho_{z--}+\rho_{z-+})+(\rho_{z+-}+\rho_{z++}) & <-> & \rho_{DM}, \\ (\rho_{z--}+\rho_{z++})+(\rho_{z+-}+\rho_{z-+}) & <-> & \rho_{DM}. \end{array} \tag{7.32}$$

That is, the density of the four species of standard matter might be related to the density of dark matter through the usual chemical potential. From this point on, when we refer to "snuark", the reader can assume that we mean standard matter snuarks unless we specify otherwise.

7.4 Binding Snuarks Together

To bind together snuarks into elementary fermions, we need to cancel off the remaining non scalar terms in Eq. (7.29). We saw how the dark matter snuark ρ_{DM} canceled these degrees of freedom the easy way. Now let us consider how to do it the hard way. We will consider the first two snuarks, that is, the $(\rho_{z--}+\rho_{z-+})$ and $(\rho_{z+-}+\rho_{z++})$. For convenience, we repeat their values here:

$$\begin{array}{lcl} \rho_{z--}+\rho_{z-+} & = & 0.25(1+\widehat{zt}-\hat{s}+\widehat{zst}), \\ \rho_{z+-}+\rho_{z++} & = & 0.25(1+\widehat{zt}+\hat{s}-\widehat{zst}). \end{array} \tag{7.33}$$

The other pair of snuarks can be treated similarly.

Since the snuarks contain a $+\widehat{zt}$ degree of freedom, to cancel this we need to add a snuark that will contain a $-\widehat{zt}$ degree of freedom. Such a snuark will travel in the $-z$ direction instead of the $+z$ direction. This is good because we will eventually need to construct stationary matter, and having components that move in opposite directions is necessary.

While a primitive idempotent contains 8 degrees of freedom, it is generated by only three choices of quantum numbers. This says that the degrees of freedom of a primitive idempotent are not independent. When we change one quantum number, in this case, $\widehat{zt}$, we change four degrees of freedom. The oriented degrees of freedom change sign, the unoriented ones do not.

Similarly, changing the sign of $\widehat{zt}$ changes the sign of two of the degrees of freedom of the snuarks. We will follow the crystallographer's convention and use $\bar{z}$ to designate a snuark oriented in the $-z$ direction. Then the two (of

interest) snuarks with -1 quantum numbers for $\widehat{zt}$ are:

$$\begin{array}{rcl} \rho_{\bar{z}--} + \rho_{\bar{z}-+} & = & 0.25(1 - \widehat{zt} - \hat{s} - \widehat{zst}), \\ \rho_{\bar{z}+-} + \rho_{\bar{z}++} & = & 0.25(1 - \widehat{zt} + \hat{s} + \widehat{zst}). \end{array} \tag{7.34}$$

We have four snuarks (i.e. the two in Eq. (7.33) and the two in Eq. (7.34)) that we want to combine into a scalar. Let's suppose that we will have A, B, C, and D portions of each. This sets up a matrix equation. Ignoring the scalar component, the requirement that the non scalar degrees of freedom add to zero gives us:

$$\begin{pmatrix} +1 & +1 & -1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.35}$$

This set of equations has only one solution, $A = B = C = D = 1$. Thus we must have all four snuarks contribute in order to get a scalar.

If we suppose that the two snuarks that travel in the $+z$ direction are present at the same time, then these two snuarks will simply add to ρ_{DM} and we will be making dark matter. Therefore, to avoid the dark matter solution, we have to assume that these snuarks are not present at the same time. In other words, we will have to include these two as a linear superposition.

To move further with this idea requires that we have a better understanding of what the mass interaction is, and how a left handed particle changes into a right handed one. This we postpone until the chapter on mass. For now, let us ignore the issue of how snuarks change themselves into different snuarks (and so change direction and various quantum numbers) and instead review the general idea of how a particle with mass is built from particles that travel at speed c.

7.5 The Feynman Checkerboard

Each snuark consists of two primitive idempotents that travel together, in the same direction at speed c. Traveling in pairs allows the elimination of the largest contribution to potential energy, the $\widehat{ixyzst}$ term. But in order to get a massive fermion that is able to remain stationary, we need to combine enough snuarks that the rest of the non scalar terms can be canceled. These cancelations will be obtained by adding together snuarks that travel in different directions and have different quantum numbers.

This concept is not entirely alien to particle physics. Richard Feynman came upon a similar idea when he was attempting to understand the nature of the Dirac equation. He was able to get the one dimensional Dirac equation by assuming that the electron moved back and forth, always at speed c, that it changed directions at random, and that each time it changed directions its wave function was multiplied by a complex constant.

The idea is known as the "Feynman Checkerboard", or sometimes chessboard, in reference to how the paths look when drawn on a $1 + 1$ space-time

diagram. The idea was originally presented as a problem in the book on the path integral by Feynman and Hibbs. A search of the internet finds quite a few articles on the subject. The first I found, not necessarily the best, are [20], [21], [22], [23]. When one tries to generalize the Feynman checkerboard model to $3+1$ dimensions, an issue that arises is that the speed of the particle making the motion wants to be $c\sqrt{3}$. This problem is discussed in Peter Plavchan's excellent term paper.[24] For the moment we will ignore this issue. We will return to it in a later chapter.

A primary objective of this paper is to derive relationships between the masses of the leptons. Particle masses are the energy of the stationary, or non moving, particle. Therefore, we will be concentrating on particles which have no net motion. To generalize to particles that are moving, one simply takes appropriate linear superpositions. This will be the subject of a book to be written by the author after the completion of the present one.

The Feynman checkerboard consists of a particle moving left and right in 1-dimensional space. In our concept of this, we will replace the left going particle with three snuarks oriented in the $+x$, $+y$, and $+z$ directions, and we will replace the right going particle with three snuarks oriented in the $-x$, $-y$, and $-z$ directions. At the points where the left going particle turns into a right going particle, we will instead have three snuarks changing form into three other snuarks. This will require considerably more machinery than that used in previous Feynman checkerboard models, but our machinery is stronger than that used previously.

In changing from one snuark form to another, spin is conserved. This gives us a hint on how to interpret the various operators as spin. Returning to the snuark quantum numbers, note that in Eq. (7.30) all four snuarks shared the same non zero quantum number for $\widehat{zt}$. We will interpret this as a precursor to spin, a sort of "pre-spin" in the $+z$ direction. Pre spin in the other three directions is therefore:

$$\begin{aligned} S_x &= \widehat{xt}, \\ S_y &= \widehat{yt}, \\ S_z &= \widehat{zt}. \end{aligned} \tag{7.36}$$

These three operators multiply to produce:

$$\begin{aligned} S_x S_y S_z &= \widehat{xt}\,\widehat{yt}\,\widehat{zt}, \\ &= \widehat{xyzt}, \end{aligned} \tag{7.37}$$

which does square to -1 as expected for the imaginary unit of the Pauli algebra.

In addition, since $-i\widehat{xyzt} = \hat{s}\ i\widehat{xyzst}$ and $\hat{s}$ and $i\widehat{xyzst}$ are two of the commuting roots of unity, then so is $-i\widehat{xyzt}$, and multiplying this by i still leaves an operator that commutes with the rest of the primitive idempotent. Therefore $\widehat{xyzt}$ will act just like an imaginary unit. One can then verify

that $S_x S_y = \widehat{xyzt} S_z$ and cyclic permutations. This completes the proof that $\{\widehat{xt}, \widehat{yt}, \widehat{zt}\}$ can be interpreted as a set of $SU(2)$ spin operators. Note that our choice of $\hat{s}$ was not required for all this to happen. The various other possible choices would have led to analogous results.

7.6 Adding Mass to the Massless

The standard model is founded on symmetry, while this book is founded on geometry. The most important symmetry is energy, and the standard model uses a Lagrangian or Hamiltonian density to specify particle interactions. The simplest way to add mass to a massless quantum field theory is simply to add a term that annihilates a right handed particle and creates a left handed particle, with the coupling constant being the mass itself, in addition, an additional term, the Hermitian conjugate, is assumed but is sometimes not mentioned. For example, with the electron, one has a mass term in the Lagrangian as follows:

$$m\bar{e}_L e_R + \text{ h.c. } = m\bar{e}_L e_R + m\bar{e}_R e_L, \tag{7.38}$$

where "h.c." stands for "Hermitian conjugate".

But the standard model uses a gauge principle to define the forces between the elementary fermions and this principle is incompatible with the above simple method of giving mass. Instead, one requires a scalar Higgs field that couples to the left and right handed fields. The gauge principle, like energy, is also a symmetry, and this is in keeping with the fact that the standard model is built on principles of symmetry. Unfortunately, some of the symmetries are not exact, so the standard model assumes that the symmetry is broken.

The method used to break symmetry requires the assumption that the vacuum of the physical world has less symmetry than the particles possess. The result is a theory that is complicated, difficult, and while it does explain many experiments, it creates large numbers of constants that can only be determined by experiment. In the operator formalism, there is no vacuum and this method of modeling the elementary particles is not available.

Since we must break with the standard model over the existence of the vacuum, we cannot logically follow the standard model in how mass is defined. Thus the natural inclination is to abandon the Higgs mechanism and pursue the simpler method of giving masses to fields, that is, with terms in the field theory such as Eq. (7.38).

It is the author's intention, after this book is complete, to write a book devoted to momentum and energy from a geometric operator point of view. However, let us briefly poach on the territory of that second book and derive the massive electron propagator from the massless propagators and the above Feynman diagrams. The justification for taking this material out of turn is that

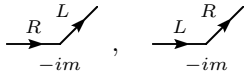
it will motivate a generalization that is necessary to explain the structure of the elementary fermions. We will also use the idea to produce formulas relating the masses of the leptons.

In the standard model, the principle of symmetry is at the foundation. In rejecting this and instead assuming that geometry is fundamental we are released from slavishly copying all the methods of deduction used in the standard model, but at the same time, we can take advantage of the well tested calculational methods of the standard model. Methods of calculation are agnostic with respect to symmetry versus geometry.

The standard model begins with a free Lagrangian. One then perturbs this Lagrangian by adding interaction terms. Calculations are simplified by the use of Feynman diagrams. In turning this method of calculation on its head, we will instead suppose that it is the Feynman diagrams themselves that are physical, and the Lagrangian just a mathematical convenience. This will free us from having to assume broken symmetries when our mathematical convenience turns out to not be perfectly symmetric.

The mass term of Eq. (7.38) has no Feynman diagrams associated with it. The mass term is part of the unperturbed field theory which can be solved exactly. However, since we have cut loose from the anchor of symmetry, we can treat the mass term as if it were a perturbation of a massless theory, and write down the Feynman diagrams that it would imply.

The two mass terms produce two Feynman diagrams. The first converts a right handed particle to left handed, the second converts the opposite way:

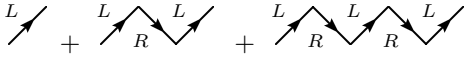


(7.39)

The above two Feynman diagrams change the propagators from left handed to right handed. If we had chosen a different signature, etc., we would have to use a different coupling constant. The $-i$ is purely mathematical as the above is not a geometric theory.

Let K_{LL} be the propagator for an electron that begins and ends its journey as left handed, K_{RL} be one that begins right handed and ends left handed, etc. We need to derive these propagators from the massless propagators and the above Feynman diagrams. Fortunately, the above Feynman diagrams are supremely simple and can be easily resummed. We begin with K_{LL}:

K_{LL} =



(7.40)

From now we will leave off the coupling constants; there is one at each vertex and they are all $-im$. The above series of Feynman diagrams sum to:

$$\begin{aligned}
K_{LL} &= \left(\tfrac{-1}{\not p}\right) + \left(\tfrac{-1}{\not p}\right)\left(\tfrac{-im}{1}\right)\left(\tfrac{-1}{\not p}\right)\left(\tfrac{-im}{1}\right)\left(\tfrac{-1}{\not p}\right) + ... \\
&= \left(\tfrac{-1}{\not p}\right)\left(1 + \left(\tfrac{-im}{1}\right)\left(\tfrac{-1}{\not p}\right)\left(\tfrac{-im}{1}\right)\left(\tfrac{-1}{\not p}\right) + ...\right) \\
&= \tfrac{-1}{\not p}\left(1 - \tfrac{m^2}{p^2} + \tfrac{m^4}{p^4} - \tfrac{m^6}{p^6} + ...\right) \\
&= \tfrac{-1}{\not p}\left(\tfrac{p^2}{p^2+m^2}\right) \\
&= \tfrac{-\not p}{p^2+m^2},
\end{aligned} \tag{7.41}$$

Similarly for the series that begin and end with right handed propagators: $K_{RR} = -\not{p}/(p^2 + m^2)$.

The Feynman diagrams that begin with left handed propagators and end with right handed propagators sum as follows:

$$
\begin{aligned}
K_{LR} &= \quad L \nearrow R \quad + \quad L \nearrow R \searrow L \nearrow R \quad + \quad L \nearrow R \searrow L \nearrow R \searrow L \nearrow R \quad ... \\
&= \left(\frac{-1}{\not{p}}\right)\left(\frac{-im}{1}\right)\left(\frac{-1}{\not{p}}\right) + \\
&\quad + \left(\frac{-1}{\not{p}}\right)\left(\frac{-im}{1}\right)\left(\frac{-1}{\not{p}}\right)\left(\frac{-im}{1}\right)\left(\frac{-1}{\not{p}}\right)\left(\frac{-im}{1}\right)\left(\frac{-1}{\not{p}}\right) + ... \\
&= \left(\frac{-im}{p^2}\right)\left(1 - \frac{m^2}{p^2} + \frac{m^4}{p^4} - \frac{m^6}{p^6} + ...\right) \\
&= \frac{-im}{p^2+m^2},
\end{aligned}
\tag{7.42}
$$

and similarly for the right to left handed propagators: $K_{RL} = -im/(p^2 + m^2)$.

In the standard model, the left and right handed electron are combined into a single particle. This single particle can be split back into its left and right handed parts using projection operators. What we've found above are the projections of the propagators in their left and right handed forms. In addition, we've computed only the electron propagators. The positron terms will be similar.

To assemble K_{LL}, K_{LR}, K_{RL} and K_{RR} into a single, massive, propagator, it is natural to choose a representation. Our choice of representation will also be used to define the gamma matrices that are implicit in $\not{p}$ and so these choices will interact. But regardless of the representation, when we define the electron propagator to be the propagator of a particle that has left and right components and includes both particle and antiparticle, we will end up with a composite (i.e. massive) propagator that includes the sums we have found:

$$K = (-\not{p} - im)/(p^2 + m^2). \tag{7.43}$$

The above can be discussed at length, with more care and detail, and the result will be the same. The point is that the massive Dirac propagator can be derived from massless Dirac propagators and the resumming of two simple Feynman diagrams.

7.7 A Composite Checkerboard

The previous two sections gave two similar methods of giving mass to massless particles. The Feynman checkerboard of Sec. (7.5) worked only in $1+1$ dimension, but it was elegant and simple. The left going and right going particles would be eigenstates of velocity which makes for a consistent and simple geometrization using Clifford algebra. The two Feynman diagrams of Sec. (7.6) worked in $3+1$ dimensions and was elegant and simple, but it assume massless Dirac propagators and these are not simple geometric objects.

In this section, we combine the two ideas into one which takes advantage of the best parts of each. We will use the Feynman checkerboard technique of

the earlier section so that we can use primitive idempotents to define the subparticles, and we will use the Feynman diagram technique of the later section so that we will turn the problem into one of a resummation.

For simplicity, we will replace the snuarks with primitive idempotents and use L and R just as labels to arbitrarily distinguish between left and right handed particles without assuming any difference in quantum numbers. This will simplify the problem considerably. We will return to the physical problem of the quarks and leptons in the next chapter.

In the first part of this chapter we showed how primitive idempotents could be bound together as snuarks. A snuark is an eigenstate of velocity and this limits how well it can minimize its energy. To reduce its energy to below the order of the Planck mass, snuarks must combine together in ways that involve velocities traveling in different directions. This is exactly the topic of the Feynman checkerboard model and the resummation of massless propagators to massive.

Pairs of primitive idempotents come equipped with a natural probability, the familiar:

$$\begin{aligned} P(A \to B) &= \langle A|B\rangle\langle B|A\rangle, \\ &= \mathrm{tr}(\rho_A\, \rho_B). \end{aligned} \tag{7.44}$$

Unfortunately for the Feynman checkerboard model, the above gives zero when ρ_A and ρ_B are oriented in opposite directions.

We could always assume that gravity is a force that violates the above assumption. But we would still only have the Dirac propagator in $1+1$ dimensions. Instead, we will assume that the above is true, that the usual $0.5(1+\cos(\theta))$ probability rule for spin-1/2 states oriented in directions separated by an angle θ, does apply.

In making this assumption, it becomes impossible for a primitive idempotent to "turn on a dime". A primitive idempotent oriented in $+z$, that is, one that carries a $+\widehat{zt}$ quantum number, cannot transform immediately into one with a $-\widehat{zt}$ quantum number. It can, however, go to $\pm\widehat{xt}$ or $\pm\widehat{yt}$.

Therefore, we will assume that the mass interaction forces the bound states of primitive idempotents to involve all six orientations:

	$\widehat{xt}$	$\widehat{yt}$	$\widehat{zt}$
ρ_x	$+2$	0	0
ρ_y	0	$+2$	0
ρ_z	0	0	$+2$
$\rho_{\bar{x}}$	-2	0	0
$\rho_{\bar{y}}$	0	-2	0
$\rho_{\bar{z}}$	0	0	-2

(7.45)

In the above we have included only the orientation quantum number. The other quantum numbers add to the confusion, but our analysis will cover them

as well (in later chapters). But our analysis will be precise for the dark matter snuarks, ρ_{DM}, which have no other quantum numbers.

We can write down a table showing the probabilities for primitive idempotent transformations from one orientation to another:

	ρ_x	ρ_y	ρ_z	$\rho_{\bar{x}}$	$\rho_{\bar{y}}$	$\rho_{\bar{z}}$
ρ_x	1	0.5	0.5	0	0.5	0.5
ρ_y	0.5	1	0.5	0.5	0	0.5
ρ_z	0.5	0.5	1	0.5	0.5	0
$\rho_{\bar{x}}$	0	0.5	0.5	1	0.5	0.5
$\rho_{\bar{y}}$	0.5	0	0.5	0.5	1	0.5
$\rho_{\bar{z}}$	0.5	0.5	0	0.5	0.5	1

(7.46)

In the above, again, we have ignored other quantum numbers.

If the mass interaction changed only the orientation quantum number of the primitive idempotents, the above table of probabilities would apply. We do not specify the mass interaction, Nature does, and it does not leave the other quantum numbers unchanged. Therefore, we have to divide the list of orientations into at least two groups. We will distinguish these with L and R, and the transition probabilities will be zero inside either one of these groups.

Arbitrarily putting the L into the $+x$, $+y$, and $+z$ orientations, the probability table looks like:

	ρ_{xL}	ρ_{yL}	ρ_{zL}	$\rho_{\bar{x}R}$	$\rho_{\bar{y}R}$	$\rho_{\bar{z}R}$
ρ_{xL}	0	0	0	0	0.5	0.5
ρ_{yL}	0	0	0	0.5	0	0.5
ρ_{zL}	0	0	0	0.5	0.5	0
$\rho_{\bar{x}R}$	0	0.5	0.5	0	0	0
$\rho_{\bar{y}R}$	0.5	0	0.5	0	0	0
$\rho_{\bar{z}R}$	0.5	0.5	0	0	0	0

(7.47)

The above can be obtained by postulating that three left handed primitive idempotents transform to right handed primitive idempotents and vice versa. This fits into a sort of Feynman checkerboard scheme, but with three particles transforming into three particles at each "m" vertex.

The above table of probabilities allows only two possible transitions each way. Looking at L to R transitions, x can either transform to $\bar{y}$ or $\bar{z}$. If x transforms to $\bar{y}$, then z can only go to $\bar{x}$ and so y can only go to $\bar{x}$. Thus there are only two Feynman diagrams that contribute to the L to R process as shown in Fig. (7.1).

For familiarity, let us write the array of transition amplitudes in the spinor form. Given an initial state $(|xL\rangle, |yL\rangle, |zL\rangle)$, the interaction transforms it to

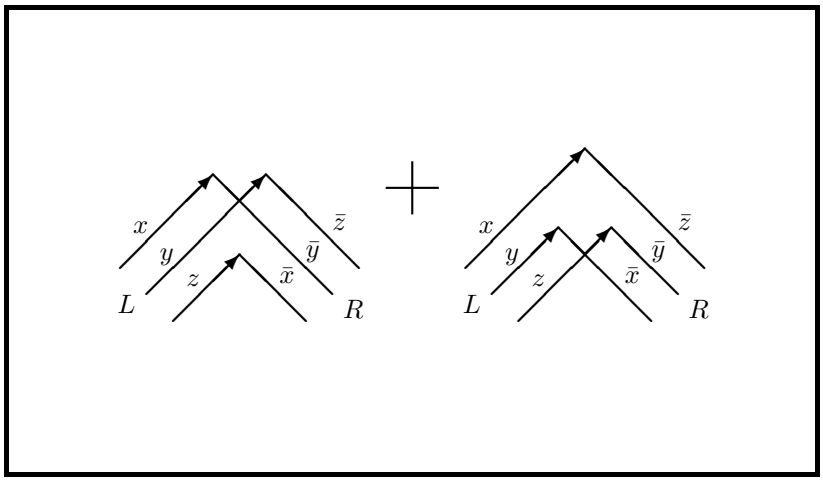


Figure 7.1: Feynman diagrams that contribute to a left handed primitive idempotent (snuark) becoming right handed.

a new state $(|\bar{x}R\rangle, |\bar{y}R\rangle, |\bar{z}R\rangle)$. The amplitudes for this are:

	xL	yL	zL
$\bar{x}R$	0	$\langle\bar{x}R\|yL\rangle$	$\langle\bar{x}R\|zL\rangle$
$\bar{y}R$	$\langle\bar{y}R\|xL\rangle$	0	$\langle\bar{y}R\|zL\rangle$
$\bar{z}R$	$\langle\bar{z}R\|xL\rangle$	$\langle\bar{z}R\|yL\rangle$	0

(7.48)

We can replace the transition amplitudes with products of primitive idempotents:

	ρ_{xL}	ρ_{yL}	ρ_{zL}
$\rho_{\bar{x}R}$	0	$\rho_{\bar{x}R}\ \rho_{yL}$	$\rho_{\bar{x}R}\ \rho_{zL}$
$\rho_{\bar{y}R}$	$\rho_{\bar{y}R}\ \rho_{xL}$	0	$\rho_{\bar{y}R}\ \rho_{zL}$
$\rho_{\bar{z}R}$	$\rho_{\bar{z}R}\ \rho_{xL}$	$\rho_{\bar{z}R}\ \rho_{yL}$	0

(7.49)

A similar set of transition amplitudes apply to the transformation from R to L:

	$\rho_{\bar{x}L}$	$\rho_{\bar{y}L}$	$\rho_{\bar{z}L}$
ρ_{xR}	0	$\rho_{xR}\ \rho_{\bar{y}L}$	$\rho_{xR}\ \rho_{\bar{z}L}$
ρ_{yR}	$\rho_{yR}\ \rho_{\bar{x}L}$	0	$\rho_{yR}\ \rho_{\bar{z}L}$
ρ_{zR}	$\rho_{zR}\ \rho_{\bar{x}L}$	$\rho_{zR}\ \rho_{\bar{y}L}$	0

(7.50)

Since we have transition amplitudes for L to R and R to L, we can combine these together to produce transition for L to R to L. The general form will be obvious if we work out a few of these by example. We will abbreviate the

intermediate states by ρ_R:

$$\begin{array}{rcl} \rho_{xL}\,\rho_R\,\rho_{xL} &=& \rho_{xL}\,\rho_{\bar{y}R}\,\rho_{xL} + \rho_{xl}\,\rho_{\bar{z}R}\,\rho_{xL}, \\ \rho_{xL}\,\rho_R\,\rho_{yL} &=& \rho_{xL}\,\rho_{\bar{z}R}\,\rho_{yL}, \\ \rho_{xL}\,\rho_R\,\rho_{zL} &=& \rho_{xL}\,\rho_{\bar{y}R}\,\rho_{zL}. \end{array} \tag{7.51}$$

The other intermediate states, for example, $\rho_{\bar{x}R}$, are eliminated by annihilation.

Since, for example, $\rho_{\bar{y}R} = (\rho_{\bar{y}R})^2$, we can rewrite the above as:

$$\begin{array}{rcl} \rho_{xL}\,\rho_R\,\rho_{xL} &=& (\rho_{xl}\,\rho_{\bar{y}R})(\rho_{\bar{y}R}\,\rho_{xL}) + (\rho_{xl}\,\rho_{\bar{z}R})(\rho_{\bar{z}R}\,\rho_{xL}), \\ \rho_{xL}\,\rho_R\rho_{yL} &=& (\rho_{xl}\,\rho_{\bar{z}R})(\rho_{\bar{z}R}\,\rho_{yL}), \\ \rho_{xL}\,\rho_R\rho_{zL} &=& (\rho_{xl}\,\rho_{\bar{y}R})(\rho_{\bar{y}R}\,\rho_{zL}). \end{array} \tag{7.52}$$

And this is just the top line in the matrix product:

$$\begin{pmatrix} 0 & \rho_{xL}\,\rho_{\bar{y}R} & \rho_{xL}\,\rho_{\bar{z}R} \\ \rho_{yL}\,\rho_{\bar{x}R} & 0 & \rho_{yL}\,\rho_{\bar{z}R} \\ \rho_{zL}\,\rho_{\bar{x}R} & \rho_{zL}\,\rho_{\bar{y}R} & 0 \end{pmatrix} \begin{pmatrix} 0 & \rho_{\bar{x}R}\rho_{yL} & \rho_{\bar{x}R}\rho_{zL} \\ \rho_{\bar{y}R}\rho_{xL} & 0 & \rho_{\bar{y}R}\rho_{zL} \\ \rho_{\bar{z}R}\rho_{xL} & \rho_{\bar{z}R}\rho_{yL} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_x\rho_{\bar{y}}\rho_x + \rho_x\rho_{\bar{z}}\rho_x & \rho_{xL}\,\rho_{\bar{z}R}\,\rho_{yL} & \rho_{xL}\,\rho_{\bar{y}R}\,\rho_{zL} \\ \rho_{yL}\,\rho_{\bar{z}R}\,\rho_{xL} & \rho_y\rho_{\bar{x}}\rho_y + \rho_y\rho_{\bar{z}}\rho_y & \rho_{yL}\,\rho_{\bar{x}R}\,\rho_{zL} \\ \rho_{zL}\,\rho_{\bar{y}R}\,\rho_{xL} & \rho_{zL}\,\rho_{\bar{x}R}\,\rho_{yL} & \rho_z\rho_{\bar{x}}\rho_z + \rho_z\rho_{\bar{y}}\rho_z \end{pmatrix} \tag{7.53}$$

where we have left off the L and R labels on the diagonal to fit to the page. Thus it is natural for us to use matrices to represent the L to R and R to L processes.

It is important to note that this sort of representation is not a general feature of Feynman diagrams. It only works here because we are assuming (by energy considerations) that all three of the initial and final states are filled by exactly one particle each.

If we represent our states $\rho_{\chi L}$ and $\rho_{\bar{\chi}R}$ in the Pauli algebra as $0.5(1+\sigma_\chi)$ and $0.5(1-\sigma_\chi)$, respectively, we can simplify the above matrix product. Recall that any product that begins and ends with $0.5(1+\sigma_\chi)$ is a real multiple of $0.5(1+\sigma_\chi)$. For example:

$$\begin{array}{l} \rho_x(a_1\hat{1} + a_x\hat{x} + a_y\hat{y} + a_z\hat{z})\rho_x, \\ = 0.5(1+\hat{x})(a_1\hat{1} + a_x\hat{x} + a_y\hat{y} + a_z\hat{z})0.5(1+\hat{x}), \\ = (a_1 + a_x)0.5(1+\hat{x}), \\ = (a_1 + a_x)\rho_x. \end{array} \tag{7.54}$$

Thus $\rho_x\rho_{\bar{y}}\rho_x = 0.5\rho_x$, and the diagonal elements in the matrix product of Eq. (7.53) reduce to give:

$$\begin{pmatrix} \rho_{xL} & \rho_{xL}\,\rho_{\bar{z}R}\,\rho_{yL} & \rho_{xL}\,\rho_{\bar{y}R}\,\rho_{zL} \\ \rho_{yL}\,\rho_{\bar{z}R}\,\rho_{xL} & \rho_{yL} & \rho_{yL}\,\rho_{\bar{x}R}\,\rho_{zL} \\ \rho_{zL}\,\rho_{\bar{y}R}\,\rho_{xL} & \rho_{zL}\,\rho_{\bar{x}R}\,\rho_{yL} & \rho_{zL} \end{pmatrix} \tag{7.55}$$

Similarly, using the methods of Sec. (2.4), the off diagonal terms reduce. For example:

$$\begin{array}{l} \rho_{xL}\,\rho_{\bar{y}R}\,\rho_{zL} \\ = 0.5(1+i)\,\rho_{xL}\,\rho_{zL}, \\ = \sqrt{0.5}\,e^{+i\pi/4}\rho_{xL}\,\rho_{zL}, \end{array} \tag{7.56}$$

where $i = \widehat{xyz}$ as before. One obtains the angle $\pm i\pi/4$ for the various off diagonal terms according to which diagonal they are in. Factoring out $\sqrt{0.5}$, the fully reduced product is:

$$\sqrt{0.5}\begin{pmatrix} \sqrt{2}\rho_{xL} & e^{-i\pi/4}\rho_{xL}\rho_{yL} & e^{+i\pi/4}\rho_{xL}\rho_{zL} \\ e^{+i\pi/4}\rho_{yL}\rho_{xL} & \sqrt{2}\rho_{yL} & e^{-i\pi/4}\rho_{yL}\rho_{zL} \\ e^{-i\pi/4}\rho_{zL}\rho_{xL} & e^{+i\pi/4}\rho_{zL}\rho_{yL} & \sqrt{2}\rho_{zL} \end{pmatrix} \tag{7.57}$$

The elements of the above matrix are Clifford algebra constants rather than the usual complex numbers.

Consider general 3×3 matrices whose elements are complex multiples of the corresponding elements of the above matrix. Such matrices are more general than complex matrices, however, since the Clifford algebraic constant is specified for each position in the matrix, the number of degrees of freedom is the same as that of complex 3×3 matrices.

The sum of two such matrices will still be of this form, so this type of matrix is closed under addition. And addition will be analogous to addition of complex matrices. While multiplication will be different from that of complex matrices, the reader can quickly verify (using the primitive idempotent product reduction equations) that matrices of this type are closed under multiplication.

What's more, matrices of this sort include a natural unit matrix:

$$\hat{1} = \begin{pmatrix} \rho_{xL} & 0 & 0 \\ 0 & \rho_{yL} & 0 \\ 0 & 0 & \rho_{zL} \end{pmatrix}. \tag{7.58}$$

therefore we have an algebra. That is, we have a set of objects which we can add and multiply, a zero (i.e. the matrix of all zeros), and a unit.

This is very significant. The arrays of amplitudes for left handed particles becoming right handed and returning to left handed form an algebra. We therefore can think of these objects as quantum states and compute their primitive idempotents using the machinery we developed in Chapter (3). This means that we can derive the structure of bound states of primitive idempotents from the structure of the primitive idempotents using the same mathematics. This is how we will derive the elementary fermions in Chapter (8). But first, let us examine exactly what this sort of sleight of hand means.

7.8 Bound State Primitive Idempotents

Our original motivation for using a density operator formalism was that the formalism worked for the elementary fermions. But now our primitive idempotents are assumed to be subparticles and the elementary fermions that make up the elementary fermions. This implies that there must exist a way for relating our composite models of fermions back to an idempotent form. In this section we address two objectives. First, we solve the primitive idempotent problem for the bound states. Second, we give an interpretation in terms of forces.

We are proposing that matrices of primitive idempotents are how one must model the natural bound states of primitive idempotents. That is, we require our bound state ρ be of the form:

$$\rho_a = \begin{pmatrix} a_{xx}\rho_x & a_{xy}\rho_x\ \rho_y & a_{xz}\rho_x\ \rho_z \\ a_{yx}\rho_y\ \rho_x & a_{yy}\rho_y & a_{yz}\rho_y\ \rho_z \\ a_{zx}\rho_z\ \rho_x & a_{zy}\rho_z\ \rho_y & a_{zz}\rho_z \end{pmatrix}, \tag{7.59}$$

where a_{jk} are complex numbers. Let ρ_b be another such matrix, with coefficients b_{jk}. To abbreviate our notation, let us designate these sorts of matrices by leaving off the ρ_χ terms inside the matrix. But to remind us that these are not the usual complex matrices, and cannot be multiplied as such, we will add a hat to the matrix as in $(\ )_\rho$. Later, when we are dealing with more complicated problems, we can add a designation outside the matrix to indicate the internal quantum numbers of the primitive idempotents.

The product $\rho_a\ \rho_b$ is of the same form, and can be computed as follows:

$$\begin{aligned} \rho_a\rho_b &= \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix}_\rho \begin{pmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix}_\rho \\ &= \begin{pmatrix} a_{xx}b_{xx} + 0.5a_{xy}b_{yx} + 0.5a_{xz}b_{zx} & \ldots & \ldots \\ a_{yx}b_{xx} + a_{yy}b_{yx} + 0.5(1+i)a_{yz}b_{zx} & \ldots & \ldots \\ a_{zx}b_{xx} + 0.5(1-i)a_{zy}b_{yx} + a_{zz}b_{zx} & \ldots & \ldots \end{pmatrix}_\rho . \end{aligned} \tag{7.60}$$

where "..." stands for six more terms similar to those shown, but which do not fit on the page. The factors of 0.5 come from reduction of products like $\rho_x\ \rho_y\ \rho_x$, and the factors of $0.5(1 \pm i)$ come from products like $\rho_x\ \rho_y\ \rho_z$.

The form of the product implies a way that we can convert our $(\ )_\rho$ matrices back and forth into the usual complex matrices. The conversion runs as follows:

$$\begin{aligned} &\begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix}_\rho \\ &\to \begin{pmatrix} a_{xx} & \sqrt{0.5}\ e^{+i\pi/12}\ a_{xy} & \sqrt{0.5}\ e^{-i\pi/12}\ a_{xz} \\ \sqrt{0.5}\ e^{-i\pi/12}\ a_{yx} & a_{yy} & \sqrt{0.5}\ e^{+i\pi/12}\ a_{yz} \\ \sqrt{0.5}\ e^{+i\pi/12}\ a_{zx} & \sqrt{0.5}\ e^{-i\pi/12}\ a_{zy} & a_{zz} \end{pmatrix} \end{aligned} \tag{7.61}$$

This conversion is linear. That is, if $(a)_\rho + (b)_\rho = (a+b)_\rho$, then $(a) + (b) = (a+b)$. And it preserves multiplication, that is if $(a)_\rho\ (b)_\rho = (ab)_\rho$, then $(a)(b) = (ab)$. Furthermore, the conversion preserves the unit matrix, and so we have a way of doing computations in $(\ )_\rho$ matrices using regular complex matrices.

In short, despite the odd form of our $(\ )_\rho$ matrices, they act in every way like the usual complex matrices. In Sec. (3.3) we analyzed the structure of the primitive idempotents of 3×3 complex matrices and that analysis can

immediately be applied here. By symmetry, we need to treat the three direction x, y, and z the same. Therefore the natural choice of primitive idempotents are the circulant.

We found the circulant 3×3 matrices in eigenvalue form as Eq. (3.41), and include them here, written out in matrix form, for convenience:

$$\begin{aligned}
\rho_1 &= \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\
\rho_\nu &= \tfrac{1}{3}\begin{pmatrix} 1 & \nu & \nu^* \\ \nu^* & 1 & \nu \\ \nu & \nu^* & 1 \end{pmatrix}, \\
\rho_{\nu^*} &= \tfrac{1}{3}\begin{pmatrix} 1 & \nu^* & \nu \\ \nu & 1 & \nu^* \\ \nu^* & \nu & 1 \end{pmatrix}.
\end{aligned} \tag{7.62}$$

We have labeled these primitive idempotents with the complex numbers making up their $(1,2)$ position. There are three primitive idempotents. Eventually we will associate these three solutions with the three generations of elementary particles, but to do this realistically we will need snuarks instead of the primitive idempotents we are discussing here.

To translate the 3×3 matrix primitive idempotents into composite primitive idempotent form, we simply use the reverse of the conversion given in Eq. (7.61). That is, we multiply the off diagonal elements by $\sqrt{2}e^{\pm i\pi/12}$. The three bound state primitive idempotents are then:

$$\begin{aligned}
\rho_1 &= \tfrac{1}{3}\begin{pmatrix} 1 & \sqrt{2}e^{-i\pi/12} & \sqrt{2}e^{+i\pi/12} \\ \sqrt{2}e^{+i\pi/12} & 1 & \sqrt{2}e^{-i\pi/12} \\ \sqrt{2}e^{-i\pi/12} & \sqrt{2}e^{+i\pi/12} & 1 \end{pmatrix}_\rho, \\
\rho_\nu &= \tfrac{1}{3}\begin{pmatrix} 1 & \sqrt{2}e^{+7i\pi/12} & \sqrt{2}e^{-7i\pi/12} \\ \sqrt{2}e^{-7i\pi/12} & 1 & \sqrt{2}e^{+7i\pi/12} \\ \sqrt{2}e^{+7i\pi/12} & \sqrt{2}e^{-7i\pi/12} & 1 \end{pmatrix}_\rho, \\
\rho_{\nu^*} &= \tfrac{1}{3}\begin{pmatrix} 1 & \sqrt{2}e^{-3i\pi/4} & \sqrt{2}e^{+3i\pi/4} \\ \sqrt{2}e^{+3i\pi/4} & 1 & \sqrt{2}e^{-3i\pi/4} \\ \sqrt{2}e^{-3i\pi/4} & \sqrt{2}e^{+3i\pi/4} & 1 \end{pmatrix}_\rho.
\end{aligned} \tag{7.63}$$

In the above, the angles are given by $-\pi/12$, $2\pi/3 - \pi/12 = 7\pi/12$, and $-2\pi/3 - \pi/12 = -3\pi/4$. That is, the rather arbitrary looking angles, $-\pi/12$, $7\pi/12$ and $-3\pi/4$ are obtained by subtracting $\pi/12$ from the three complex cube roots of unity. Thus we can solve the bound state primitive idempotent problem.

We've been using matrices of products of primitive idempotents, as in Eq. (7.59) to represent bound states. The diagonal elements are complex multiples of primitive idempotents, ρ_x, ρ_y, and ρ_z. However, in Eq. (7.63) we see that the only "complex multiples" of the diagonal elements are $1/3$. Thus the diagonal elements are treated equally. This is exactly what we expect if these three primitive idempotents are combined into a bound state.

It is natural to expect that there should only be one bound state for the three primitive idempotents; instead we've found three. But the difference between these three solutions is not in the diagonal elements, but instead in the off diagonal.

The off diagonal terms are products of two primitive idempotents, for example $\rho_x\ \rho_y$. In the operator formalism, products of two primitive idempotents are what we use to represent a process that converts a particle of one type into another.

In the standard model, one expects that the conversion of a fermion from one type to another is accompanied by the creation or annihilation of a gauge boson. Thus we can think of the off diagonal elements as having something to do with a particle interaction.

In replacing the Feynman checkerboard interaction with a matrix, we enforced the requirement that the number of particles of each type was conserved. That is, we began with one ρ_{xL}, one ρ_{yL}, and one ρ_{zL}, and these became exactly one $\rho_{\bar{x}R}$, one $\rho_{\bar{y}R}$, and one $\rho_{\bar{z}R}$. One normally associates a conservation like this with an exchange force.

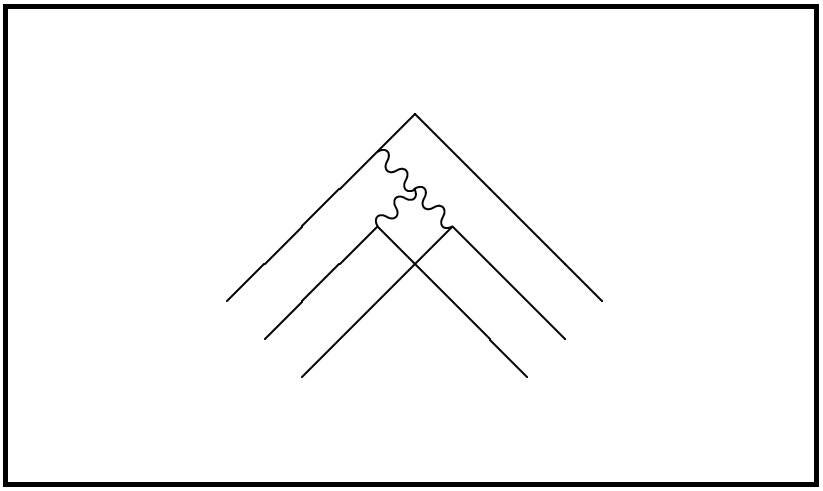


Figure 7.2: The mass interaction as an exchange of three "gauge bosons".

When the standard model deals with gauge bosons, it must define a propagator for the exchange particle. In our case, the three particles are transformed at the same point in space. Thus there are no propagators for these bosons, as they do not propagate anywhere. We can draw gauge bosons as in Fig. (7.2).

This gives us an interpretation for the bound state primitive idempotents. The diagonal elements correspond to the "valence primitive idempotents", and the off diagonal elements are the "primitive idempotent sea". In this interpretation, the three generations of fermions correspond to the same valence primitive idempotents (and therefore have the same quantum numbers), but correspond to different levels of excitation in the sea.

Chapter 8

The Zoo

An' I sign for four-pound-ten a month and save the money clear,
An' I am in charge of the lower deck, an' I never lose a steer;
An' I believe in Almighty God an' preach His Gospel here.

The skippers say I'm crazy, but I can prove 'em wrong,
For I am in charge of the lower deck with all that doth belong—
Which they would not give to a lunatic, and the competition so strong!

THE PREVIOUS CHAPTER showed how it is natural to suppose that primitive idempotents combine so as to cancel their strongest degrees of freedom, and the resulting "snuarks" naturally form into the same doublet and dual singlet form seen in the elementary fermions. Then we showed how to make idempotency calculations with collections of primitive idempotents that are bound together in a manner similar to how mass binds left and right handed particles. The result was that the bound states exhibited three levels of excitation of the sea, and we claimed that these could be thought of as the three generations of elementary fermions.

For simplicity, these two ideas, binding primitive idempotents into snuarks and collecting primitive idempotents into bound states, were applied separately. In this chapter, we apply them simultaneously, and show how the result can be interpreted as the three generations of elementary fermions.

8.1 The Mass Interaction

In Chapter (7), we associated the mass interaction with three primitive idempotents exchanging gauge bosons. For example three left handed primitive idempotents can be changed into right handed primitive idempotents. Our expectation is that the left and right handed composite states will give a sum that has no non scalar parts left. The resulting purely scalar sum will define the mass.

Looking at this from the point of view of potential energy, we expect the left and right handed states to have non scalar parts that cancel. That is, the non scalar part of the left handed state needs to be the negative of the non scalar part of the right handed state. But this implies that the gauge bosons have completely negated the non scalar part of the left handed state.

Since the non scalar parts of a quantum state are the parts that have Planck energies, we expect this process to leave a gauge boson with energy around the Planck energy. But this seems rather unphysical.

To explain how the mass interaction can change the non scalar parts of quantum states, we need to reexamine three facts from the previous chapter: First, we found that a natural potential energy needs to ignore, or almost ignore, the scalar part of a Clifford algebra element. Second, we found that the natural mass interaction needs to change the sign of Clifford algebra primitive idempotents as in $0.5(1 + \widehat{zt})$ to $0.5(1 - \widehat{zt})$. Third, we found that primitive idempotents will bind in groups of three, and when doing this, there will be three different choices for the sea that binds them together.

This book has been devoted to the discrete symmetries of elementary particles or quantum states. In so concentrating, we have ignored the fact that elementary particles form interference patterns. This suggests that instead of thinking of our elementary particles as idempotents such as $0.5(1 + \widehat{zt})$, we should instead think of them as having some sort of phase, for example, $e^{i\omega t}0.5(1+\widehat{zt})$. In other words, we should think of $-0.5(1+\widehat{zt})$ as representing the same elementary particle as $0.5(1 + \widehat{zt})$.

Since the natural potential energy needs to almost ignore the scalar part of a Clifford algebra primitive idempotent, this suggests that we should represent the mass interaction as a negating of the scalar part of a quantum object. This will give something that is not normalized, so we then negate the whole object. We will call this operation M. For example:

$$\begin{aligned} M(0.5(1 + \widehat{zt})) &= -(0.5(-1 + \widehat{zt})), \\ &= +0.5(1 - \widehat{zt}). \end{aligned} \tag{8.1}$$

In general, M will negate all but the scalar part of an element. In the above example, M maps an idempotent to an idempotent, but we are more interested in what M does to primitive idempotents.

Note that our mass interaction, M, preserves our potential energy, V. That is, since our potential energy is a sum of multiples of squares, changing the signs of the vector part does not change the potential energy:

$$V(M(\rho)) = V(\rho). \tag{8.2}$$

Any modification which consists of changing the signs of some set of components of the Clifford algebra will preserve potential energy, but the one we are

using minimizes the change by assuming that the change is to the scalar part only.

To see what M does to primitive idempotents, let's choose primitive roots of unity $\hat{s}$, $\widehat{zt}$, and $\widehat{ixyzst}$. A typical primitive idempotent looks like:

$$\begin{aligned} \rho_{z++} &= 0.125(1+\widehat{zt})(1+\hat{s})(1+\widehat{ixyzst}), \\ &= 0.125(1+\widehat{zt}+\hat{s}+\widehat{ixyzst}-\widehat{zst}-\widehat{ixys}-\widehat{ixyzt}-\widehat{ixy}). \end{aligned} \tag{8.3}$$

Applying M to this primitive idempotent negates all but the scalar term giving:

$$M(\rho_{z++}) = 0.125(1-\widehat{zt}-\hat{s}-\widehat{ixyzst}+\widehat{zst}+\widehat{ixys}+\widehat{ixyzt}+\widehat{ixy}). \tag{8.4}$$

The above is not a primitive idempotent. The first four terms look like the first four terms of $\rho_{\bar{z}--}$, but the next three terms have the wrong signs:

$$\begin{aligned} \rho_{z--} &= 0.125(1-\widehat{zt})(1-\hat{s})(1-\widehat{ixyzst}), \\ &= 0.125(1-\widehat{zt}-\hat{s}-\widehat{ixyzst}-\widehat{zst}-\widehat{ixys}-\widehat{ixyzt}+\widehat{ixy}), \\ &\neq M(\rho_{z++}). \end{aligned} \tag{8.5}$$

Thus M does not map primitive idempotents to primitive idempotents. The result is not a quantum state, it is instead a mixture.

In quantum mechanics, when a state propagates to a mixture, a measurement of the quantum numbers of the state gives a result proportional to the transition probabilities between the mixed state and the various states contributing to the mixture. This is exactly what we want in a mass interaction. But instead of applying M to a primitive idempotent, we need to apply it to bound states of the sort we found in the previous chapter.

The mass interaction defined in Eq. (8.5) will mix states completely. We will associate the usual elementary particles with mixtures that are preserved by this interaction. Rather than solve this problem immediately, let's take a look at how the solution will look.

If the mixture were equally distributed over all the states, then all the quantum numbers would be zero and the particle would be dark matter. For standard matter, the solutions of interest will be the ones that are not evenly balanced. We will now assume that the states of interest are categorized by having one snuark represented more than the others. Thus, for the purposes of looking at quantum numbers, we can look at the quantum numbers of the snuarks themselves.

8.2 Snuark Antiparticles

In the standard model, the quantum numbers of antiparticles are the negatives of the quantum numbers of the particles. This is somewhat at odds with what

we know about the quantum numbers of primitive idempotents. For example, a typical primitive idempotent is:

$$\begin{aligned}\rho_{z-+} &= 0.125(1+\widehat{zt})(1-\hat{s})(1+\widehat{ixyzst}),\\ &= 0.125(1+\widehat{zt}-\hat{s}-\widehat{ixyzst}+\widehat{zst}-\widehat{ixys}+\widehat{ixyzt}+\widehat{ixy}),\end{aligned} \tag{8.6}$$

and it's eigenvalues can be read off the signs of the second line, that is, the seven (non trivial) quantum numbers are $+1$, -1, $+1$, $+1$, -1, $+1$, and -1.

If we are to define the antiparticle as the particle that has the negatives of all these quantum numbers, the particle will therefore be:

$$\rho_{z^-_{-+}} = 0.125(1-\widehat{zt}+\hat{s}-\widehat{ixyzst}-\widehat{zst}+\widehat{ixys}-\widehat{ixyzt}-\widehat{ixy}). \tag{8.7}$$

But the above does not satisfy the idempotency equation $\rho^2=\rho$ and is therefore not an idempotent, primitive or otherwise.

The difficulty arises from the fact that quantum numbers for primitive idempotents are inherently multiplicative. As soon as our Clifford algebra is complicated enough to include two commuting roots of unity, the product of those two quantum numbers will not be negated when the two quantum numbers are negated. For the moment, we will ignore this problem, which is similar to what happens with the mass interaction.

In defining an antiparticle as having negated quantum numbers, if we are to assume that they are primitive idempotents we have to define which three of the seven possible quantum numbers are to be negated. The set of seven commuting roots of unity for our example are:

$$\begin{array}{cccc}\hat{s}, & \widehat{ixy}, & \widehat{zt}, & \\ \widehat{zst}, & \widehat{ixys}, & \widehat{ixyzt}, & \widehat{ixyzst}\end{array} \tag{8.8}$$

The quantum numbers of an observed particle are the quantum numbers that have not been canceled in the binding together of the particle. Therefore, it makes sense to choose the three of these seven that have the least energy. The potential energy of those in the top row are v_s, v_s^2, and v_sv_t. None of these are smaller than the potential energies of those in the second row, $v_s^2v_t$, v_s^3, $v_s^3v_t$, and $v_s^4v_t$. Therefore the top row are the commuting roots of unity with minimum potential energy.

Negating the $\widehat{zt}$ quantum number means that the direction of travel of the antiparticle is the opposite of the particle. This is compatible with the idea that the antiparticles correspond to particles traveling backwards in time. Therefore, we will classify the antiparticles that carry a -1 quantum number for $\widehat{zt}$ as traveling in the same direction as the particles with a $+1$ quantum number.

The question then arises whether a primitive idempotent and the antiparticle of a primitive idempotent can combine into a snuark. While we ignored this possibility in the previous section, we now rectify this.

In contrast to the previous chapter, by taking into account antiparticles there are now eight primitive idempotents, and we label them by $(\widehat{zt}, \hat{s}, \widehat{ixy})$:

	$\widehat{zt}$	$\hat{s}$	$\widehat{ixy}$	$\widehat{zst}$	$\widehat{ixys}$	$\widehat{ixyzt}$	$\widehat{ixyzst}$
$\rho_{\bar{z}--}$	$-$	$-$	$-$	$-$	$+$	$+$	$-$
$\rho_{\bar{z}-+}$	$-$	$-$	$+$	$-$	$-$	$-$	$+$
$\rho_{\bar{z}+-}$	$-$	$+$	$-$	$+$	$-$	$+$	$+$
$\rho_{\bar{z}++}$	$-$	$+$	$+$	$+$	$+$	$-$	$-$
ρ_{z--}	$+$	$-$	$-$	$+$	$+$	$-$	$+$
ρ_{z-+}	$+$	$-$	$+$	$+$	$-$	$+$	$-$
ρ_{z+-}	$+$	$+$	$-$	$-$	$-$	$-$	$-$
ρ_{z++}	$+$	$+$	$+$	$-$	$+$	$+$	$+$

(8.9)

In the above we have represented the quantum numbers of ± 1 by $\pm$.

There are four primitive idempotents with positive $\widehat{ixyzst}$ quantum numbers and four with negative. This gives sixteen possible snuarks that cancel $\widehat{ixyzst}$. After $\widehat{ixyzst}$, the next largest commuting root of unity is $\widehat{ixyzt}$. Of the sixteen snuarks, eight of these also have zero for $\widehat{ixyzt}$. Leaving off the two columns of zero for $\widehat{ixyzst}$ and $\widehat{ixyzt}$ quantum numbers, these eight snuarks are:

	$\hat{s}$	$\widehat{zt}$	$\widehat{zst}$	$\widehat{ixy}$	$\widehat{ixys}$
$\rho_{z+-,z++}$	$+$	$+$	$-$	0	0
$\rho_{\bar{z}+-,\bar{z}++}$	$+$	$-$	$+$	0	0
$\rho_{z--,z-+}$	$-$	$+$	$+$	0	0
$\rho_{\bar{z}--,\bar{z}-+}$	$-$	$-$	$-$	0	0
$\rho_{z+-,\bar{z}+-}$	$+$	0	0	$-$	$-$
$\rho_{z++,\bar{z}++}$	$+$	0	0	$+$	$+$
$\rho_{z--,\bar{z}--}$	$-$	0	0	$-$	$+$
$\rho_{z-+,\bar{z}-+}$	$-$	0	0	$+$	$-$

(8.10)

In the above we represent quantum numbers of ± 2 by $\pm$.

While there are five non zero quantum numbers in the snuarks of Eq. (8.9), there are only three that are independent. We can graph them using $\hat{s}$, $\widehat{zt}$ and $\widehat{ixy}$. See Fig. (8.1), and compare with Fig. (6.5). The primitive idempotents form a cube with edges of length 2. The eight snuarks, by contrast, form the corners of a block with two edges of length $2\sqrt{2}$ and one edge of length 4, at least in the above choice of quantum numbers. Since quantum numbers can be scaled, the snuark quantum numbers cannot be distinguished from a cube.

8.3 Quarks

Fig. (8.1) gives the quantum numbers for one snuark. Following the modified Feynman checkerboard model of the previous chapter, we expect the chiral

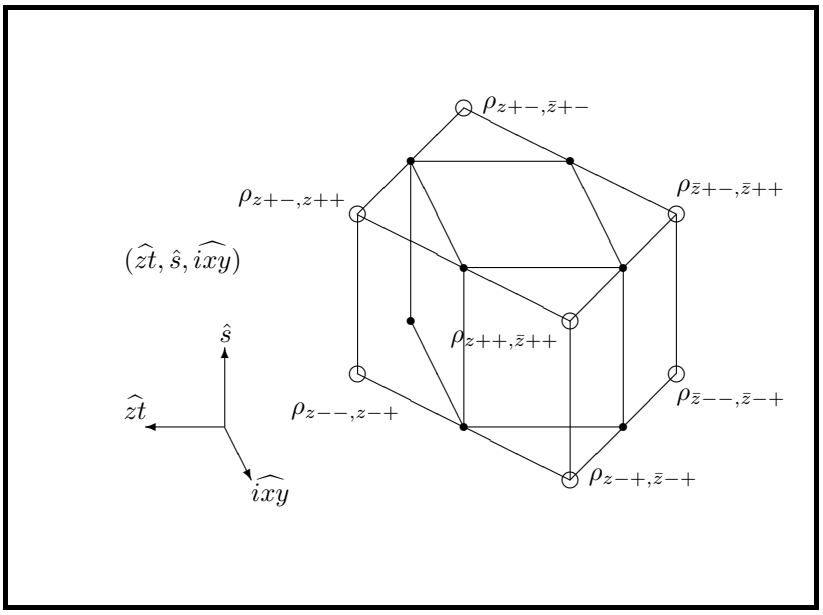


Figure 8.1: Quantum number plot of the eight snuarks with zero $\widehat{ixyzst}$ and $\widehat{ixyzt}$. The primitive idempotents are marked with small solid circles and form a cube aligned with the axes.

fermions to require three snuarks each. In combining snuarks, it is natural to group three snuarks of the same sort, but with different orientations. For example, one could combine $\rho_{x+-,\bar{x}+-}$, $\rho_{y+-,\bar{y}+-}$, and $\rho_{z+-,\bar{z}+-}$. We will associate the leptons with these sorts of combinations.

According to our modified Feynman checkerboard model, three snuarks differing only by orientation will combine to produce a chiral fermion that will be able to produce a particle whose potential energy is purely scalar. What would be the effect of a substitution of one of the snuarks? The result will be that the potential energy of the combination will no longer be purely scalar. Since we have a model of the potential energy, we can minimize the change in potential energy caused by the substitution of a snuark.

As an example, let suppose that our three snuarks are of type $\rho_{*+-,*++}$ where $*$ represents x, y, and z, and that this snuark contributes to a fermion which is purely scalar. Then we have

$$\rho_{x+-,x++} + \rho_{y+-,y++} + \rho_{z+-,z++} + \quad \text{RH} = \quad \text{scalar}, \tag{8.11}$$

where "RH" stands for the same sum but for the right handed particle. (At the moment we ignore the difficulties in defining how a mass interaction is mapped to the primitive idempotents and snuarks.)

The snuarks have five nonzero quantum numbers, $\hat{s}$, $\widehat{zt}$, $\widehat{ixy}$, $\widehat{zst}$ and $\widehat{ixys}$. The potential energy of these commuting roots of unity are v_s, $v_s v_t$, v_s^2, $v_s^2 v_t$, and v_s^3. Of these, the two highest potential energies are $v_s^2 v_t$ and v_s^3, corresponding to $\widehat{zst}$ and $\widehat{ixys}$. Following the same energy principle we've been using for combining primitive idempotents into snuarks, we assume that when snuarks substitute for one another, these two degrees of freedom must be conserved.

Referring to Eq. (8.10), we see that requiring that these quantum numbers not be changed in a substitution, groups the snuarks into four groups of two:

	$\hat{s}$	$\widehat{zt}$	$\widehat{zst}$	$\widehat{ixy}$	$\widehat{ixys}$
$\rho_{z+-,z++}$	+	+	−	0	0
$\rho_{\bar{z}--,\bar{z}-+}$	−	−	−	0	0
$\rho_{\bar{z}+-,\bar{z}++}$	+	−	+	0	0
$\rho_{z--,z-+}$	−	+	+	0	0
$\rho_{z+-,\bar{z}+-}$	+	0	0	−	−
$\rho_{z-+,\bar{z}-+}$	−	0	0	+	−
$\rho_{z--,\bar{z}--}$	−	0	0	−	+
$\rho_{z++,\bar{z}++}$	+	0	0	+	+

(8.12)

When a snuark is substituted, the quantum numbers of the resulting quark will be intermediate between the quantum numbers of the corresponding leptons.

Referring again to Fig. (8.1), our substitution rule allows snuarks to pair if they are on opposite corners. Rather than graph them in this manner, let us replace our $\widehat{zt}$ and $\widehat{ixy}$ axes with $\widehat{zst}$ and $\widehat{ixys}$. That is, instead of graphing $\hat{s}$ with the next two smallest potential degrees of freedom, we will instead graph $\hat{s}$ with the two largest potential degrees of freedom.

This sort of change to the choice of independent quantum numbers used in plotting the particles is done for convenience only. The student should remember that there is no physical 3−dimensional space for these graphs.

By our energy principle, the large degrees of freedom that will be conserved in substitution, and by graphing with these degrees of freedom, the quarks will form columns between the leptons. See Fig. (8.2).

Each quark comes in three colors. In our model, this corresponds to the fact that there are three ways the substitution can take place. In the example of substitutions of snuarks of type $\rho_{*+-,*++}$, the alternatives are snuarks of type $\rho_{\bar{*}--,\bar{*}-+}$. Composites of this type form the leftmost column of Fig. (8.2).

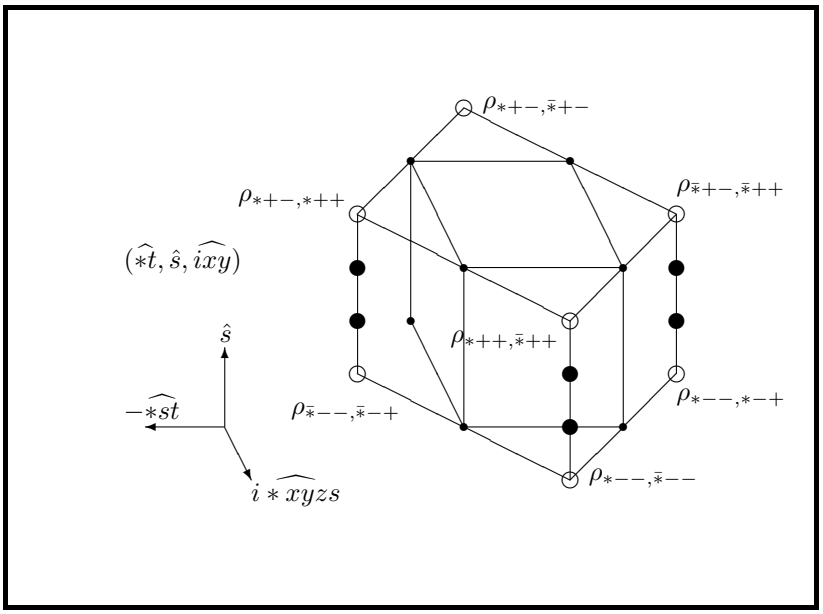


Figure 8.2: Quantum number plot of the eight leptons and sixteen quarks in snuark form relative to the primitive idempotents. Primitive idempotents are small filled circles, leptons are large hollow circles and quarks ($\times 3$) are large filled circles.

The two leptons and six quarks, with their (non zero) quantum numbers, are:

			$\hat{s}$	$\widehat{zt}$	$\widehat{zst}$
$\rho_{x+-,x++}$	$\rho_{y+-,y++}$	$\rho_{z+-,z++}$	$+6$	$+6$	-6
$\rho_{\bar{x}--,\bar{x}-+}$	$\rho_{y+-,y++}$	$\rho_{z+-,z++}$	$+2$	$+2$	-6
$\rho_{x+-,x++}$	$\rho_{\bar{y}--,\bar{y}-+}$	$\rho_{z+-,z++}$	$+2$	$+2$	-6
$\rho_{x+-,x++}$	$\rho_{y+-,y++}$	$\rho_{\bar{z}--,\bar{z}-+}$	$+2$	$+2$	-6
$\rho_{x+-,x++}$	$\rho_{\bar{y}--,\bar{y}-+}$	$\rho_{\bar{z}--,\bar{z}-+}$	-2	-2	-6
$\rho_{\bar{x}--,\bar{x}-+}$	$\rho_{y+-,y++}$	$\rho_{\bar{z}--,\bar{z}-+}$	-2	-2	-6
$\rho_{\bar{x}--,\bar{x}-+}$	$\rho_{\bar{y}--,\bar{y}-+}$	$\rho_{z+-,z++}$	-2	-2	-6
$\rho_{\bar{x}--,\bar{x}-+}$	$\rho_{\bar{y}--,\bar{y}-+}$	$\rho_{\bar{z}--,\bar{z}-+}$	-6	-6	-6

(8.13)

The leptons are the top and bottom particles.

There are a total of 8 leptons and 24 quarks whose quantum numbers are shown in Fig. (8.2). By the same method shown in Section (7.8), each of these 32 composites comes in three choices of sea, and these are the three generations. For the remainder of this chapter we will ignore the details of the generations.

The strongest force between the 32 leptons and quarks arises from their largest potential degree of freedom. This force corresponds to the "strong" force of the standard model. This is the uncanceled vector potential of the quarks. The leptons add to scalars and so do not partake in the strong force.

Since the strong force is associated with a non scalar, its strength is of the magnitude of the Planck energy, and therefore quarks will bind together so strongly that ordinary energies will be unable to separate them.

To determine how the strong force acts between quarks, we need to compare the degrees of freedom of a snuark with the snuark it can substitute for. That is, we must compute the differences between the pairs of snuarks of Eq. (8.12):

	$\hat{s}$	$\widehat{zt}$	$\widehat{zst}$	$\widehat{ixy}$	$\widehat{ixys}$
$\rho_{z--,z-+} - \rho_{\bar{z}+-,\bar{z}++}$	-4	$+4$	0	0	0
$\rho_{z-+,\bar{z}-+} - \rho_{z+-,\bar{z}+-}$	-4	0	0	$+4$	0
$\rho_{z+-,z++} - \rho_{\bar{z}--,\bar{z}-+}$	$+4$	$+4$	0	0	0
$\rho_{z++,\bar{z}++} - \rho_{z--,\bar{z}--}$	$+4$	0	0	$+4$	0

, (8.14)

and similarly for x and y. The negatives of these are also available.

We will define the color charges as red, green, and blue, and, in reference to a lepton made from snuarks traveling in the $+x$, $+y$, and $+z$ directions, we will associate red with a substitution of the $+x$ snuark, green with a substitution of the $+y$ snuark, and blue with a substitution of the $+z$ snuark.

To make an elementary fermion requires that we give the left handed as well as the right handed part. Therefore, to compute the potential energy between a collection of quarks we will have to add together the right and left handed portions. At this time we have not yet fully explained the mass interaction. However, peeking ahead, we will find that the mass interaction puts the snuarks (or the equivalents to snuarks) into pairs, and we have conveniently arranged for the snuark differences listed in Eq. (8.14) to be arranged in pairs. That is, they are listed in left handed / right handed pairs.

Eq. (8.14) are written in terms of $+z$ snuarks, so they define the blue quantum numbers. Since the left handed state has to accompany the right handed state, we have to add these together by pairs to define the blue quantum numbers. The two pairs give different sums, B_1 and B_2 as follows:

	$\hat{s}$	$\widehat{zt}$	$\widehat{zst}$	$\widehat{ixy}$	$\widehat{ixys}$
B_1	-8	$+4$	0	$+4$	0
$\bar{B}_1$	-8	-4	0	-4	0
B_2	$+8$	$+4$	0	$+4$	0
$\bar{B}_2$	$+8$	-4	0	-4	0

. (8.15)

For example, $B_1 = (\rho_{z--,z-+} - \rho_{\bar{z}+-,\bar{z}++}) + (\rho_{z-+,\bar{z}-+} - \rho_{z+-,\bar{z}+-})$, while B_2 is the sum of the last two lines of Eq. (8.14).

In computing B_1 and B_2 as differences between snuarks we took the sum over all degrees of freedom. The $\hat{s}$ degree of freedom has the lowest potential energy, v_s, and is small compared to that of the strong force, $v_s^2 v_t$ or v_s^3. In the standard model, the forces are divided according to their strength. This

suggests that we need to think of the blue quantum number as the average of B_1 and B_2.

With this interpretation, we ignore the $\hat{s}$ degree of freedom. The red, green, and blue quantum numbers are:

	$\widehat{xt}$	$\widehat{iyz}$	$\widehat{yt}$	$-\widehat{ixz}$	$\widehat{zt}$	$\widehat{ixy}$
R	$+4$	$+4$	0	0	0	0
G	0	0	$+4$	$+4$	0	0
B	0	0	0	0	$+4$	$+4$
$\bar{R}$	-4	-4	0	0	0	0
$\bar{G}$	0	0	-4	-4	0	0
$\bar{B}$	0	0	0	0	-4	-4

. (not quite right) (8.16)

To get a scalar sum, we can add R to $\bar{R}$, or the same with green and blue. Collections of colored particles that satisfy this requirement are allowed in the standard model, but the standard model also allows sums that have equal amounts of red, green, and blue.

In Section (8.1), we made the assumption that "the states of interest are categorized by having one snuark represented more than the others." In other words, we approximated the solution by representing solutions that were predominantly oriented in one direction by solutions that were oriented only in that direction. This approximation prevents our having colored states add to a colorless state.

To allow colorless states to appear as a sum of the R, G and B states, we need to replace the 0s of Eq. (8.16) with ± 1 and divide the ± 4 values by two:

	$\widehat{xt}$	$\widehat{iyz}$	$\widehat{yt}$	$-\widehat{ixz}$	$\widehat{zt}$	$\widehat{ixy}$
R	$+2$	$+2$	-1	-1	-1	-1
G	-1	-1	$+2$	$+2$	-1	-1
B	-1	-1	-1	-1	$+2$	$+2$
$\bar{R}$	-2	-2	$+1$	$+1$	$+1$	$+1$
$\bar{G}$	$+1$	$+1$	-2	-2	$+1$	$+1$
$\bar{B}$	$+1$	$+1$	$+1$	$+1$	-2	-2

. (8.17)

These quantum numbers will give the correct behavior for quarks. We will justify these assumptions later in the book.

8.4 Spinor and Operator Symmetry

It is time to make contact with the standard model. This book is written from a geometric point of view. Our objective is to understand the elementary particles as geometric objects, and to maintain as close a relationship to the physics of the particles as possible. Our method is a "top down" approach. We begin with geometry, that is, a Clifford algebra. We make the assumption that the elementary particles are primitive idempotents, and we then find out as much as we can about primitive idempotents of the Clifford algebra.

By contrast, the standard model is built from a "bottom up" procedure. One looks to experiment to find things that are conserved (for example, momentum, energy, electric charge), or what is the same thing, a symmetry, and one incorporates these experimental observations into a model of the elementary particles that obey them.

For example, Einstein showed that experiments could be interpreted as implying that the results of an experiment could never depend on the absolute motion of the experiment. The modeler then builds this observation into the model of the elementary particles. This is an indirect way of understanding the physical world and it is subject to several complications and limitations.

Newton developed a method of modeling physical situations that was very direct and had clear and simple physical interpretations. Particles were assumed to be connected to other particles by forces. Forces cause acceleration of the particles according to the mass of the particle:

$$F_x = md^2x/dt^2. \tag{8.18}$$

Mass, m, is simply a constant of proportionality between acceleration in a direction, d^2x/dt^2, and force in that direction, F_x. Since force is arbitrarily defined, so is mass, and one can always redefine them.

When one applies Newton's equations to a collection of N particles, one obtains a set of N differential equations. To predict the motion of the system, one inserts an initial state into the differential equations and solves them. Initial states were position and velocity. To find the final state, for example, the state at time T, one takes the solution of the differential equation at time T. The differential equation also provided solutions for the intermediate states.

In the 19th century, physicists and mathematicians found that they could rewrite Newton's laws as a variational principle. Instead of writing down N differential equations, they could instead write down a "Lagrangian", a real valued function of the positions and velocities of the N particles. They next suppose that the motion of the system maximizes (or minimizes) this function. Newton's differential equations can then be derived by the Euler-Lagrange equations.

The Lagrangian formalism had various advantages over Newton's equations. Being only a single function, it was simpler than the full set of differential equations. One could model constrained systems without having to keep track of the constraint forces. But for our story, the most important advantage was that it was very easy to incorporate symmetry into the Lagrangian function.

In the 1920s, it was realized that since elementary particles can have such small masses, the initial and final states of an experiment can never be determined with perfect precision. Thus the model of an experiment would have to be statistical in nature. To achieve this, the positions and momentums (i.e. real functions of time) of classical mechanics were replaced by position and momentum wave functions (i.e. probability densities, or functions of space and time).

But the probability densities of classical statistical mechanics turned out to be inefficient at solving quantum problems. While they could be made to work (i.e. Bohmian mechanics), a much simpler technique was to use complex valued functions of space and time, that is, wave functions. The method could be written in Lagrangian form, and therefore models could be easily made that would incorporate symmetries observed in experiment.

Einstein's relativity defines a symmetry, Lorentz invariance. Since linear equations are easier to solve than nonlinear ones, Dirac searched for, and found, a linear equation that would satisfy Lorentz invariance. The Dirac equation was wildly successful at modeling the electron (and lies at the heart of the Clifford algebra used in this book).

With the success of the Dirac equation, physics was hooked. Experimental results were examined for other symmetries, and as these were discovered, they were incorporated into quantum mechanics. Most physicists believed that nature was, at heart, a collection of symmetries.

If one postulates that a differential equation satisfies a simple symmetry principle, then one can obtain some knowledge about the solutions of the differential equation even if one does not know the differential equation, cannot write it down, and cannot solve it exactly. In this case, one typically ends up with some arbitrary constants that depend on the details of the differential equations, but one cannot solve the differential equations by symmetry alone. And this is precisely the problem with the standard model today, too many arbitrary constants.

On the other hand, if one postulates that a differential equation is of a simple form, then one can solve the differential equation directly. Thus the assumption that equations of physics are simple is far more powerful than the assumption that the equations of physics have simple symmetries.

What's more, physicists found experimental situations where symmetry was only approximate. This should have been a clue that symmetry was a blind alley, but since great advances had been made using symmetry, physicists refused to abandon it.

When physicists attempt to extend the standard model, they typically do this by postulating yet more symmetry. The most important thing for them is how this more general symmetry is broken to the observed symmetry, $SU(3) \times SU(2) \times U(1)$. The particular representations of that symmetry, that is, the

elementary particles themselves, are treated secondarily. It is the symmetry that the usual approach to physics treats primarily.

Since we are deriving the standard model in the reverse direction, from the bottom up, our method must instead concentrate on the representatives of the symmetry. We then show that these objects possess the symmetry given them in the standard model. As a first step, we've already seen in Section (7.8) that our bound states will appear in three varieties that we interpret as the generations.

In Section (8.3), we showed that our bound states appeared in the correct numbers to give 24 quarks and 8 leptons, under the interpretation that the leptons were pure mixtures and the quarks were not. And we showed that the rule that quarks only appear in colorless mixtures was plausible. The leptons were singlets for color, while the quarks were triplets. This singlet / triplet structure reminds one of the $SU(3)$ structure of the leptons and quarks, but the symmetry of our model is simpler than $SU(3)$.

Our representation was based on a choice of three perpendicular vectors in 3 spatial dimensions. We used the example of $\hat{x}$, $\hat{y}$, and $\hat{z}$ as indicating those vectors. The symmetry of this sort of choice is not $SU(3)$, but instead is only $SO(3)$, the symmetry of proper rotations of 3-dimensional real space. Both $SU(3)$ and $SO(3)$ are proper rotations, that is, they both are connected components that include the identity rotation.

The quantum states of the standard model are represented by spinors, while we are using density operators. Spinors possess an arbitrary complex phase, while density operators do not. This accounts for the difference in symmetry between $SU(3)$ and $SO(3)$. That is, $SU(3)$ preserves the length of 3-dimensional complex vectors, while $SO(3)$ preserves the length of 3-dimensional real vectors. Thus the symmetry of our quarks matches the $SU(3)$ symmetry of the standard model quarks, after taking into account the difference between spinors and density operators.

The $SU(2)$ symmetry of the $SU(3) \times SU(2) \times U(1)$ standard model is also called "weak isospin". The Pauli algebra also corresponds to an $SU(3)$ symmetry, but it is distinct from weak isospin in more than just application.

The Pauli algebra contains three generators, $\hat{x}$, $\hat{y}$ and $\hat{z}$. Given any unit vector (u_x, u_y, u_z), we can define the operator $u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$ and define quantum states that are (spin-1/2) eigenstates of this operator. And all these different states can be produced in the lab.

By contrast, weak isospin defines the relationship between left handed and right handed states. The left handed states show up in doublet representations

while the right handed states are singlets. If one were to imagine these states defined with a different axis (as can be had with the Pauli algebra), we could have a state that is mixed between electron and neutrino. These sorts of things are not observed in the lab. In standard quantum mechanics they are excluded by superselection rules. Consequently, in showing that our particles have an $SU(2)$ symmetry equivalent to that of the standard model, we do not have to produce a full set of generators for this $SU(2)$ symmetry.

Consider the Clifford algebra with only one dimension, say $\hat{x}'$. There are two eigenstates, $0.5(1+\hat{x}')$ and $0.5(1-\hat{x}')$. We can think of these two states as being an $SU(2)$ doublet similar to the $SU(2)$ doublet of weak isospin. We can always increase our Clifford algebra by adding two more vectors, $\hat{y}'$, and $\hat{z}'$, to obtain a true $SU(2)$ representation, but in doing this our $0.5(1\pm\hat{x}')$ eigenstates will not change.

In Section (7.3) we showed that it was natural that two primitive idempotents that differ in their $i\widehat{xyzst}$ quantum numbers, but have the same velocity, would bind together. The quantum numbers of the bound states, Eq. (7.30), we repeat here for convenience:

	1	$\widehat{zt}$	$\widehat{ixys}$	$i\widehat{xyzst}$	$\hat{s}$	$\widehat{zst}$	$i\widehat{xyzt}$	$\widehat{ixy}$
$\rho_{z--}+\rho_{z-+}$	2	2	0	0	-2	$+2$	0	0
$\rho_{z+-}+\rho_{z++}$	2	2	0	0	$+2$	-2	0	0
$\rho_{z--}+\rho_{z++}$	2	2	0	0	0	0	-2	-2
$\rho_{z+-}+\rho_{z-+}$	2	2	0	0	0	0	$+2$	$+2$

(8.19)

The four rightmost columns all show the same structure. There are two snuarks with quantum number 0, and a pair with quantum number ± 2. Any of these has the same structure as weak isospin.

The reader may note that these four quantum numbers are oriented. They depend on the velocity direction that the snuark, but since any given snuark does have a velocity, we do not have a choice for how these four quantum numbers are oriented. For example, if we suppose $\widehat{zst}$ represents weak isospin, the first two states form the doublet, and the second two states are singlets.

With this interpretation, it is satisfying to see that weak isospin does have a full $SU(2)$ symmetry, that is, continuing our example, the set $\{\widehat{xst}, \widehat{yst}, \widehat{zst}\}$ does form a set of generators for an $SU(2)$. (I.e., like the vector generators of a Clifford algebra, each squares to $+1$ and they anticommute.) We can thus interpret the superselection rule as arising from the fact that weak isospin is only defined relative to the chiral states, which are massless, travel at c, and have an orientation.

We have treated the standard model symmetries $SU(3)$ and $SU(2)$ differently in that we took advantage of the difference between spinors and density operators to explain the difference between $SU(3)$ and $SO(3)$, but we did not

rely on this sort of argument with $SU(2)$. In analogy with the $SU(3)$ situation, we can reduce $SU(2)$ to a real rotation that preserves real vectors of length 2, that is, $SO(2)$. There is then only one generator, and the representations include the ones observed in the snuarks.

Finally, in moving from spinors to density operators, the arbitrary complex phase of the $U(1)$ symmetry of the standard model is eliminated. Thus we see that the symmetry of the 32 states described in Section (8.3) matches that of the standard model. It remains, however, to check that the whole collection of states have quantum numbers that are compatible overall with those of the standard model.

8.5 Weak Hypercharge and Isospin

It remains to compare the quantum numbers of one generation of the standard model fermions with the quantum numbers of the quarks and leptons made from snuarks. The snuark version of quarks are built from a combination of primitive idempotent particles and primitive idempotent antiparticles, so in comparing quantum numbers we must compare the full set of particles and antiparticles.

To make the comparison, let us graph the quarks and leptons according to their weak isospin and weak hypercharge quantum numbers. The quantum numbers of the left handed antiparticles are simply the negatives of the quantum numbers of the right handed particles, and the same for the right handed antiparticles and left handed particles. See Table (8.3).

	t_0	t_3
e_R	-1	0
e_L	$-1/2$	$-1/2$
ν_R	0	0
ν_L	$-1/2$	$+1/2$
d_R	$-1/3$	0
d_L	$+1/6$	$-1/2$
u_R	$+2/3$	0
u_L	$+1/6$	$+1/2$

	t_0	t_3
$\bar{e}_L$	$+1$	0
$\bar{e}_R$	$+1/2$	$+1/2$
$\bar{\nu}_L$	0	0
$\bar{\nu}_R$	$+1/2$	$-1/2$
$\bar{d}_L$	$+1/3$	0
$\bar{d}_R$	$-1/6$	$+1/2$
$\bar{u}_L$	$-2/3$	0
$\bar{u}_R$	$-1/6$	$-1/2$

Figure 8.3: Weak hypercharge (t_0) and weak isospin (t_3) quantum numbers for the first generation particles and antiparticles.

Weak hypercharge and weak isospin are sufficient to distinguish all the

elementary fermions except for the right handed neutrino, ν_R and the left handed antineutrino, $\bar{\nu_L}$. In the original "standard model", the neutrino was assumed massless and these two states were not included. The density operator model automatically includes these states and so we have added them to the standard model for comparison.

In extending the standard model to include neutrino masses, there are several possibilities, including the usual Dirac mass term, or a Majorana mass term. More complicated ways of giving mass to the neutrino involves the addition of "sterile" neutrinos. In the next chapter we will discuss this at great length, for now, we will simply add these two neutrinos as if we were adding a Dirac mass term.

Other quantum numbers for the fermions are sometimes given. The weak isospin quantum number shown is weak isospin in the 3rd, direction, t_3. Some authors add a quantum number t to distinguish the doublets, $\{e_L, \nu_L, d_L, u_L\}$, (which take $t = 1/2$) from the singlets, $\{e_R, \nu_R, d_R, u_R\}$ (which take $t = 0$).

The standard model particles are defined according to how they interact with each other. The electromagnetic coupling is called the electric charge, Q, and is given by the sum of the weak hypercharge and weak isospin:

$$Q = t_3 + t_0. \tag{8.20}$$

An analogous coupling for the weak force is called [25, Table 6.2] the "neutral charge", Q'. To obtain the neutral charge from weak hypercharge and weak isospin one requires the Weinberg angle θ_W:

$$Q' = t_3 \cot(\theta_W) - t_0 \tan(\theta_W), \tag{8.21}$$

where $\sin^2(\theta_W)$ is approximately $1/4$. The transformation from weak hypercharge to weak isospin is a linear one, so we can plot them together, see Fig. (8.4).

The similarity between Fig. (8.2) and Fig. (8.4) show that the quantum numbers of bound states of snuarks come in a 3-dimensional pattern (four parallel edges of a cube) that is easy to match to the observed quantum numbers of the elementary fermions.

There are still two difficulties. First, we did not solve the mass interaction completely. Instead, we approximated its solutions by assuming that the solution states could be approximated by the snuarks themselves. Second, there are a number of different ways of associating the axes of the two figures. In addition to the obvious possibility of rotating the two cubes, one can also imagine that we could choose different commuting roots of unity and thereby could transform the snuark cube.

In solving these issues, we will finally obtain complete geometric models for the elementary particles. This will allow us to begin computing attributes

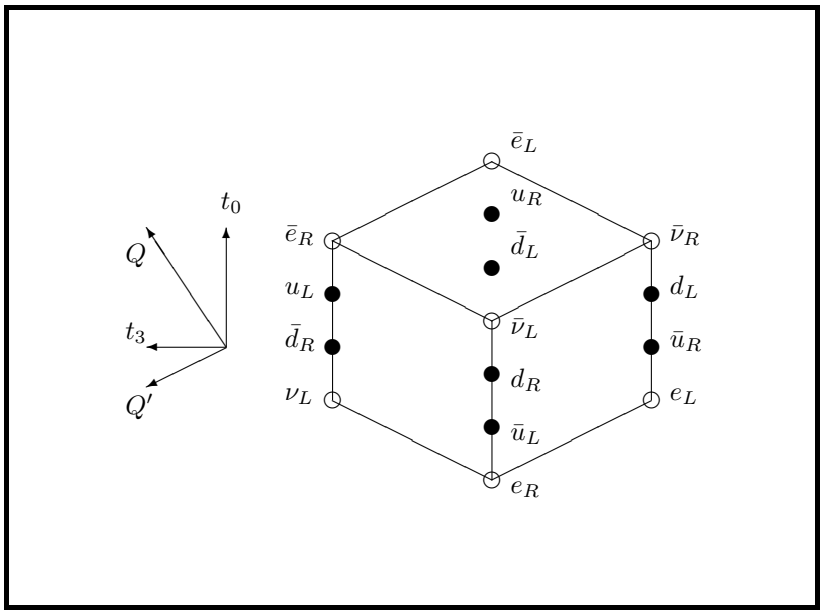


Figure 8.4: Weak hypercharge, t_0, and weak isospin, t_3, quantum numbers plotted for the first generation standard model quantum states. Leptons are hollow circles and quarks ($\times 3$) are filled circles. Electric charge, Q, and neutral charge, Q' also shown.

of the elementary particles from first principles. This will be the topic of the next chapter. The first attributes we will solve for will be the masses of the elementary fermions, which can be thought of as the mass charges.

8.6 Antiparticles and the Arrow of Time

Following Feynman, we define antiparticles as particles that travel backwards in time. However, given a pair of states, one a particle, the other an antiparticle, we have no way of determining which is the one that travels backwards in time and which travels forwards. A particle and its antiparticle form a related set, but we cannot logically apply the label "antiparticle" to one of them rather than the other. If we were able to distinguish the true direction of travel in time of a quantum state, forwards in time and therefore a particle, versus backwards in time and therefore an antiparticle, we would have a way of defining the "arrow of time" in quantum mechanics.

But the standard model includes no such arrow of time. Instead, we define antiparticles by arbitrarily assuming that the states we see in everyday matter are made up of particles, and their complementary states are antiparticles. In

fact, there are bound states of particles with antiparticles, for example, pions:

$$\begin{aligned} \pi^+ &= u\bar{d}, \\ \pi^- &= d\bar{u}. \end{aligned} \tag{8.22}$$

Clearly these mixed bound states are inherently neither particles nor antiparticles. We have the choice of defining them either way. Looked at this way, particle / antiparticle works best as a relative relationship between two states rather than a description that can be absolutely applied to a single state.

Our primitive idempotents are eigenstates of velocity, and we defined their antiparticles as particles that carried the same quantum number for velocity, but travel in the opposite direction. As such, the primitive idempotent model does have an arrow of time, one that is consistent across all the primitive idempotents, if not the elementary particles.

Of course in fitting the model to experiment we will have two choices, and the arrow of time will melt away, but unlike the standard model, in the density operator model we can naturally extend the particle / antiparticle relationship to all the elementary particles.

We therefore examine the particle / antiparticle content of the snuark model of the elementary fermions. Repeating Eq. (8.10), there are eight snuarks:

	$\hat{s}$	$\widehat{zt}$	$\widehat{zst}$	$\widehat{ixy}$	$\widehat{ixys}$
$\rho_{z+-,z++}$	+	+	−	0	0
$\rho_{\bar{z}+-,\bar{z}++}$	+	−	+	0	0
$\rho_{z--,z-+}$	−	+	+	0	0
$\rho_{\bar{z}--,\bar{z}-+}$	−	−	−	0	0
$\rho_{z+-,\bar{z}+-}$	+	0	0	−	−
$\rho_{z++,\bar{z}++}$	+	0	0	+	+
$\rho_{z--,\bar{z}--}$	−	0	0	−	+
$\rho_{z-+,\bar{z}-+}$	−	0	0	+	−

. (8.23)

Of these, two are pure antiparticle, $\rho_{\bar{z}--,\bar{z}-+}$ and $\rho_{\bar{z}+-,\bar{z}++}$, two are pure particle, $\rho_{z+-,z++}$ and $\rho_{z--,z-+}$, and the remaining four are mixed.

Returning to the snuark and quark quantum number plot Fig. (8.2), there are two lepton states that are pure particle, two that are pure antiparticle, and the remaining four are mixed. The four mixed leptons are either the weak isospin singlets or the doublets. All the quarks are mixed. Comparing with Fig. (8.4), it appears that the mass interaction converts pure snuarks to mixed snuarks. But we must again stress that these purported lepton states were made under the assumption that the true leptons were snuarks, an assumption that we know is incompatible with our assumed gravitational interaction.

Chapter 9

Mass

> Up along the hostile mountains, where the hair-poised snow-slide shivers—
> Down and through the big fat marshes that the virgin ore-bed stains,
> Till I heard the mile-wide mutterings of unimagined rivers
> And beyond the nameless timber saw illimitable plains!

WE HAVE POSTULATED a fundamental force that binds primitive idempotents. We have shown that the elementary fermions of the standard model have a structure similar to what you would expect from a complete cancelation of the non scalar parts of this force. We have guessed at a form for the mass interaction. We have speculated that the measured masses of the elementary particles is given by the scalar remnant of the fundamental force after the vector portion of that force is completely canceled. We have assumed that the observed leptons correspond to pure snuarks, and we have seen that the resulting set of states are similar in nature to the quarks and leptons of the standard model. It remains to improve this assumption and to calculate the observed elementary fermion masses.

9.1 Statistical Mixtures

Suppose that we have a set of N probabilities p_n (i.e. numbers between 0 and 1) that add up to unity, and I have N primitive idempotents ρ_n, defined according to the same commuting roots of unity. Then we have:

$$\begin{array}{rcl} p_1 + p_2 + ... + p_n & = & 1, \\ \rho_1 + \rho_2 + ... + \rho_n & = & \hat{1}. \end{array} \tag{9.1}$$

We define a statistical mixture as a sum over the primitive idempotents:

$$\rho = p_1\rho_1 + p_2\rho_2 + ... + p_N\rho_N. \tag{9.2}$$

The signs of the vector parts of the primitive idempotents depend on the quantum numbers of the particular primitive idempotent. Only the scalar part

always has the same sign. For the $C(4,1)$ Clifford algebra, there are 3 commuting roots of unity and therefore the scalar part is always $2^{-3} = 1/8$. Since the probabilities add up to one, the scalar part of a statistical mixture will also be 1/8.

In Section Eq. (8.1) we postulated that the mass interaction, M, has the effect of negating the signs of all the non scalar portions of a primitive idempotent:

$$\begin{aligned} M(A) &= M(a_1\hat{1} + a_x\hat{x} + a_y\hat{y} + ... + i\widehat{xyzst}), \\ &= a_1\hat{1} - a_x\hat{x} - a_y\hat{y} - ... - i\widehat{xyzst}. \end{aligned} \tag{9.3}$$

We noted that M has the disquieting effect of not preserving primitive idempotents. That is, if ρ_χ is a primitive idempotent, $M(\rho_\chi)$ is not. Instead, it is a more general operator.

However, M preserves the scalar part of a primitive idempotent:

$$\langle M(a_1\hat{1} + a_x\hat{x} + ...)\rangle_0 = a_1 = \langle (a_1\hat{1} + a_x\hat{x} + ...)\rangle_0. \tag{9.4}$$

and consequently we can hope that $M(\rho_\chi)$ can be interpreted as a function that maps statistical mixtures to statistical mixtures.

In defining the modification to the Feynman checkerboard, we assumed that the transition amplitudes were given by products of the primitive idempotents. We then assembled the amplitudes into a 3×3 matrix of amplitudes and associated the bound states with the matrices that were primitive idempotents in the algebra of all such matrices.

To make the Feynman checkerboard generalization work with our mass interaction, we need to replace its primitive idempotents with statistical mixtures. To see how this works, let's consider products of two different statistical mixtures, ρ and ρ'. Each of these will have N probabilities and primitive idempotents, but following our earlier efforts in the Feynman checkerboard matrices, we do not assume that they use the same sets of primitive idempotents:

$$\begin{aligned} \rho &= p_1\rho_1 + p_2\rho_2 + ... + p_N\rho_N, \\ \rho' &= p'_1\rho'_1 + p'_2\rho'_2 + ... + p'_N\rho'_N. \end{aligned} \tag{9.5}$$

The contribution to the transition amplitude, $\rho_m\ \rho'_n$, needs to be multiplied by the probability that both of these are the actual primitive idempotent. We assume that the probabilities are independent, so this probability is just the product of the probabilities, $p_n p'_m$. Summing up over all possible choices of the two primitive idempotents, the statistical mixture of the transition amplitude is simply:

$$\begin{aligned} \textstyle\sum_m \sum_n (p_m p'_n\ \rho_m\ \rho'_n) &= (\textstyle\sum_m p_m\rho_m)(\sum_n p'_n\rho'_n), \\ &= \rho\ \rho'. \end{aligned} \tag{9.6}$$

Thus the machinery for making computations with statistical mixtures of primitive idempotents is similar to the machinery we've already been using with the primitive idempotents themselves.

If $M(\rho_{z++})$ is a statistical mixture, then we can write it as a sum over primitive idempotents. To compute it as a sum over primitive idempotents oriented in the z direction, the probabilities are computed as the scalar parts of $8M(\rho_{z++})\ \rho_{\pm z\pm\pm}$. The 8 comes from the conversion from the "trace" function used in matrices to the "scalar part" function used in geometry. We first write out $M(\rho_{z++})$:

$$\begin{aligned} M(\rho_{z++}) &= M(0.125(1+\widehat{zt}+\hat{s}+i\widehat{xyzst}-\widehat{zst}-i\widehat{xys}-i\widehat{xyzt}-\widehat{ixy})), \\ &= 0.125(1-\widehat{zt}-\hat{s}-i\widehat{xyzst}+\widehat{zst}+i\widehat{xys}+i\widehat{xyzt}+\widehat{ixy}). \end{aligned} \tag{9.7}$$

Then, after a certain amount of calculation, the transition probabilities are:

$$\begin{array}{llll} p_{z++}=-0.75, & p_{z+-}=+0.25, & p_{z-+}=+0.25, & p_{z--}=+0.25, \\ p_{\bar{z}++}=+0.25, & p_{\bar{z}+-}=+0.25, & p_{\bar{z}-+}=+0.25, & p_{\bar{z}--}=+0.25. \end{array} \tag{9.8}$$

Note that while the probabilities do add to unity, one of them, the one corresponding to the probability of the mass interaction leaving the primitive idempotent unchanged, is negative. This indicates the mass interaction does not map the primitive idempotent ρ_{z++} to a statistical mixture, which should not be very surprising. The result is obviously general, none of the primitive idempotents $\rho_{z\chi\chi}$ will be mapped to a statistical mixture.

We can also compute the transition probabilities for $M(\rho_{z++})$ to primitive idempotents oriented in the x direction. We obtain:

$$\begin{array}{llll} p_{z++}=-0.25, & p_{z+-}=+0.25, & p_{z-+}=+0.25, & p_{z--}=+0.25, \\ p_{\bar{z}++}=-0.25, & p_{\bar{z}+-}=+0.25, & p_{\bar{z}-+}=+0.25, & p_{\bar{z}--}=+0.25. \end{array} \tag{9.9}$$

Again, the transition probabilities sum to unity, but two of them are negative. The negative ones are the ones that leave the last two quantum numbers unchanged.

We next look at the action of M on snuarks. Let us repeat the calculation with $M(\rho_{z+-,z++})$. Referring to Eq. (8.12) for the snuark values:

$$\begin{aligned} M(\rho_{z+-,z++}) &= M(0.25(1+\widehat{zt}+\hat{s}-\widehat{zst}), \\ &= 0.25(1-\widehat{zt}-\hat{s}+\widehat{zst}). \end{aligned} \tag{9.10}$$

Since snuarks are combinations of two primitive idempotents, the scaling factor is changed. To see what it is, note that $(\rho_{z+-,z++})^2 = \rho_{z+-,z++}$, so the transition probability should be unity. But the scalar part of $\rho_{z+-,z++}$ is 0.25, so the scaling factor needs to be 4. Accordingly, compute the scalar part of

$4M(\rho_{z+-,z++})\ \rho_{\chi,\chi}$:

$$\begin{array}{llll} p_{z+-,z++} = -0.5, & p_{\bar{z}--,\bar{z}-+} = +0.5, & p_{\bar{z}+-,\bar{z}++} = +0.5, & p_{z--,z-+} = +0.5, \\ p_{z+-,\bar{z}+-} = +0.5, & p_{z-+,\bar{z}-+} = +0.5, & p_{z--,\bar{z}--} = +0.5, & p_{z++,\bar{z}++} = +0.5. \end{array} \tag{9.11}$$

The probabilities sum two 2 because the snuarks doubly represent the number of degrees of freedom. And again we have a negative probability for the transition from number. Also, since our snuarks are degenerate, that is, since they share degrees of freedom, the probabilities sum to more than 1.

Snuarks with different orientation can share only two degrees of freedom, $\hat{1}$ and $\hat{s}$. Therefore, the transition probabilities will all be $(1-1+0+0)/4 = 0$ or $(1+1+0+0)/4 = 0.5$, depending on the eigenvalue of $\hat{s}$. Referring to Eq. (8.12), the transition probabilities are:

$$\begin{array}{llll} p_{x+-,x++} = 0, & p_{\bar{x}--,\bar{x}-+} = +0.5, & p_{\bar{x}+-,\bar{x}++} = 0, & p_{x--,x-+} = +0.5, \\ p_{x+-,\bar{x}+-} = 0, & p_{x-+,\bar{x}-+} = +0.5, & p_{x--,\bar{x}--} = +0.5, & p_{x++,\bar{x}++} = 0. \end{array} \tag{9.12}$$

Note that none of the above are negative, and that M will map a snuark to a statistical mixture of four snuarks. And that statistical mixture is a little special; it is equal for all four values.

We represent Stern-Gerlach filters with projection operators, and the key attribute of Stern-Gerlach filters is that after they measure an attribute of a particle they preserve that attribute. This is the physical reason why it is natural for us to use primitive idempotents to represent the elementary particles.

However, the idempotency equation, ρ^2 is never equal to ρ for a statistical mixture that is not pure. On the other hand, we can imagine a more general Stern-Gerlach filter that would preserve a statistical mixture. As before, we will represent such a Stern-Gerlach filter with the same mathematical object that represents the statistical mixture, ρ itself.

Given a complete set of 8 primitive idempotents, the most complete mixture one can make is to take 0.125 of each:

$$\rho_c = 0.125(\rho_1 + \rho_2 + ... + \rho_8) = 0.125\hat{1}. \tag{9.13}$$

Since this is a mixture, it is no longer idempotent, but it is suspiciously close:

$$\rho_c^2 = 0.125\rho_c. \tag{9.14}$$

That is, ρ_c is just an eighth of the unity operator, which is idempotent, but not a primitive idempotent. This suggests that a natural extension of the idempotency equation to statistical mixtures is to allow

$$\rho^2 = k\rho, \tag{9.15}$$

where k is a real constant. The value $0.125\hat{1}$ is a pure scalar and therefore has a very low potential energy. This suggests that we should consider the potential energies of statistical mixtures.

In making calculations with primitive idempotents, one of the most useful techniques we used was the fact that any product that begins with a primitive idempotent and ends with another, will be a complex multiple of the two primitive idempotents on the ends. For example:

$$\rho_m \, X \, \rho'_n = x_{mn} \, \rho_m \, \rho'_n, \tag{9.16}$$

where X is any operator, and x_{mn} is a complex number. From a physical point of view, this corresponds to putting X between two Stern-Gerlach filters. Because the filters allow only a single state to exit, the stuff in the middle, X, can only have the effect of decreasing the amplitude, or changing its phase.

9.2 The Koide Relation

In 1982, Yoshio Koide postulated [26] a relationship between the masses of the charged leptons.

Chapter 10

Cosmic Haze

The Spirit gripped him by the hair, and sun by sun they fell
Till they came to the belt of Naughty Stars that rim the mouth of Hell.
The first are red with pride and wrath, the next are white with pain,
But the third are black with clinkered sin that cannot burn again:
They may hold their path, they may leave their path, with never a soul to mark,
They may burn or freeze, but they must not cease in the Scorn of the Outer Dark.
The Wind that blows between the Worlds, it nipped him to the bone,
And he yearned to the flare of Hell-gate there as the light of his own hearth-stone.

WHILE SNUARKS AND anions are too high energy for man to create them with current technology, nature has access to higher energies and we can suppose that they were present at the big bang, and that they may be let loose by the extreme conditions at black holes.

Chapter 11

Conclusion

> Small mirth was in the making—now
> I lift the cloth that cloaks the clay,
> And, wearied, at thy feet I lay
> My wares, ere I go forth to sell.
> The long bazar will praise, but thou—
> Heart of my heart—have I done well?

Perhaps we've applied density operator formalism with too much enthusiasm and have over reached a calculation or two. Apologies to the reader. Certainly we've added this chapter mostly to provide an opportunity to quote from yet one more Kipling poem.

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