Density Operators, Spinors, and the Particle Generations

Carl Brannen

Redmond, Washington

Abstract. In quantum information theory, qubits are typically represented by Pauli spinors. In this paper we discuss alternative representations of qubits, as spinors or density operators, and with or without a matrix representation of the spin operators selected, and show that an operator representation with no matrices provides an elegant, powerful, easy to use, and geometric formalism. We present techniques for calculation in geometric form.

We apply Julian Schwinger's "measurement algebra" to the problem of representing qubits. Following Schwinger, we show that the spinors can be represented in the operator algebra by choosing an arbitrary "vacuum" state. Representing these states in geometric form, non intuitive spinor behavior such as the multiplication by -1 that results from a 2π rotation, are explained.

Extending qubits to more complicated Clifford algebras, we speculate that the structure of the elementary fermions of the standard model implies one hidden dimension and a preon structure. An analysis of the preon structure derives the Koide relation for the charged lepton masses. We extend this relation to the neutrinos and predict their masses.

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1. DENSITY MATRICES

The fundamental fermions of the standard model are spin-1/2 objects, and this paper will concern itself with the representation of these objects. The standard model is written in terms of spinors, but spin-1/2 quantum states can also be represented by density matrices. This paper concerns itself with density operators (i.e. density matrices, but without choosing any particular matrix representation), of spin-1/2 quantum states. We will find that density operators provide a convenient, simple, and easy to interpret geometric description of spin-1/2 quantum states.

1.1. Notation

We will write our operators as small, hatted Roman letters with, and their eigenvalues will be the corresponding small letters. For example, the operator $\hat{x}$ will have eigenvalues such as x, x_1, x_2. We will use small Roman letters to designate real numbers. With this notation, an eigenvalue implies the operator it is associated with. Our density operators will be written as ρ_χ, where χ gives the eigenvalue(s). Spinors will be written in bra ket notation.

For example, $\langle b_1|$, $|b_1\rangle$, and ρ_{b1} are a bra, the associated ket, and the density operator representing the same quantum state. The density operator is usually defined from the

bra and ket:

$$\rho_{b1} = |b_1\rangle \langle b_1|. \tag{1}$$

We will show how to reverse this process, and to define bras and kets from density operators. Consequently, we do not treat the above as a definition of the density operator. It is simply a way to define a density operator formalism in terms of a spinor formalism. The density operator formalism can stand on its own without any need for a spinor formalism.

Our concern is with the spin degrees of freedom. Spin can be measured in various directions, and we will define our directions by unit vectors written as small Roman letters with an arrow symbol. For example, $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors that define the $\hat{u}$, $\hat{v}$, and $\hat{w}$ spin-1/2 operators in their respective directions. Vectors have components, for example, $\vec{u} = (u_x, u_y, u_z)$.

Of particular importance are the spin operators in the $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ directions. In the usual quantum mechanics formalism, these spin operators are represented by matrices and are written σ_x, σ_y, and σ_z. This paper is concerned with geometry, not matrix representations, and in order to make it clear that we are not using a particular representation of the Pauli algebra, we will instead designate these three spin operators as $\hat{x}$, $\hat{y}$ and $\hat{z}$. The spin operator in an arbitrary direction, for example $\vec{u}$, can be written as a sum over $\hat{x}$, $\hat{y}$, and $\hat{z}$:

$$\hat{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}. \tag{2}$$

While our concern is with geometry rather than matrix representations, our intended audience is very familiar with the Pauli spin matrices, so we will use them for illustration. The matrices are defined as:

$$\begin{aligned}
\hat{x} \equiv \sigma_x &= \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \\
\hat{y} \equiv \sigma_y &= \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \\
\hat{z} \equiv \sigma_z &= \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned} \tag{3}$$

But it is important to note that our definitions will not require the Pauli spin matrices or any other representation, and will be defined only from the operators.

The various representations of the quantum state that have eigenvalues of +1/2 for spin in the z direction are $|z\rangle$, $\langle z|$, and ρ_z. Note that the Pauli operators have integer eigenvalues, ± 1, rather than $\pm 1/2$. We will work with the Pauli operators. If you prefer to work with spin operators, simply multiply the Pauli operators by $1/2$. The eigenvalue equations these states satisfy are:

$$\begin{aligned}
\hat{z}\ |z\rangle &= +1|z\rangle &&= |z\rangle, \\
\hat{z}\ \rho_z &= +1\rho_z &&= \rho_z, \\
\langle z|\ \hat{z} &= +1\langle z| &&= \langle z|, \\
\rho_z\ \hat{z} &= +1\rho_z &&= \rho_z.
\end{aligned} \tag{4}$$

Also important are the quantum states that have spin eigenvalues of $-1/2$ and therefore Pauli eigenvalues of -1. We will designate these eigenvalues with "bar" notation. For example,

$$\begin{array}{rcrcl} \hat{z}\ |\bar{z}\rangle & = & -1|\bar{z}\rangle & = & -|\bar{z}\rangle, \\ \hat{z}\ \rho_{\bar{z}} & = & -1\rho_{\bar{z}} & = & -\rho\bar{z}, \\ \langle\bar{z}|\ \hat{z} & = & -1\langle\bar{z}| & = & -\langle\bar{z}|, \\ \rho_{\bar{z}}\ \hat{z} & = & -1\rho_{\bar{z}} & = & -\rho\bar{z}, \end{array} \tag{5}$$

are the eigenvalue equations for the $-1/2$ states.

1.2. The Spin Algebra

Given a spin operator $\hat{u}$ and a real number k, we can multiply these together to define a new operator, $k\hat{u}$. The new operator, $k\hat{u}$ is not a spin operator unless $k = \pm 1$, in which case it is either unchanged, or it is $\hat{\bar{u}}$, the spin operator pointing in the opposite direction as $\hat{u}$. That is, $-\hat{u} = \hat{\bar{u}}$.

The spin operators form an algebra. That is, an addition and multiplication among the operators is defined. Given two spin operators, $\hat{u}$, and $\hat{v}$, the following are also operators (though not necessarily spin operators):

$$\begin{array}{c} \hat{u}+\hat{v}, \\ \hat{u}\hat{v} \end{array} \tag{6}$$

Some of these sums and products are particularly simple. For example, $\hat{u}\,\hat{\bar{u}} = \hat{u}+\hat{\bar{u}} = 0$, and $\hat{u}\,\hat{u} = \hat{u}+\hat{\bar{u}} = 1$. If we used the spin operators instead of the Pauli operators, we would have inconvenient factors of $1/2$ appearing here.

To simplify products of spin operators we need to define the products of $\hat{x}$, $\hat{y}$, and $\hat{z}$. We will do this by giving what the mathematicians call a "presentation". That is, we will define the rules that may be used to simplify products:

$$\begin{array}{rcl} \hat{x}\hat{y} & = & -\hat{y}\hat{x}, \\ \hat{y}\hat{z} & = & -\hat{z}\hat{y}, \\ \hat{z}\hat{x} & = & -\hat{x}\hat{z}, \\ \hat{x}^2 = \hat{y}^2 = \hat{z}^2 & = & 1. \end{array} \tag{7}$$

These are the rules for a Clifford algebra of signature $+++$. We will follow the mathematical terminology and call $\hat{x}$, $\hat{y}$, and $\hat{z}$ "canonical basis vectors".

The first three of the above rules tell us how to anticommute distinct canonical basis vectors. Using these rules, any product of the canonical basis vectors can be manipulated down to one of the following 16 forms:

$$\pm\hat{x}^{n_x}\,\hat{y}^{n_y}\,\hat{z}^{n_z}, \tag{8}$$

where n_x, n_y, and n_z are each 0 or 1. The $\pm$ comes from the anticommutation rules. Ignoring the factor of ± 1, there are eight possible products in the above. We will call

these the "canonical basis elements":

$$\begin{array}{cccc} 1, & \hat{x}, & \hat{y}, & \hat{z}, \\ \hat{x}\hat{y}\hat{z}, & \hat{y}\hat{z}, & \hat{x}\hat{z}, & \widehat{xy}. \end{array} \tag{9}$$

The algebra of operators can be thought of as a vector space over the real numbers, and the canonical basis elements give a basis set for that vector space.

Of the canonical basis elements, $\hat{x}\hat{y}\hat{z}$ is special in that it commutes with all three canonical basis vectors, and therefore commutes with all elements of the algebra. Furthermore, it squares to -1, and consequently can be thought of as a sort of imaginary unit. This suggests that the operator algebra can be thought of as a complex vector space over the four basis elements $\{1, \hat{x}, \hat{y}, \hat{z}\}$, rather than a real vector space over the eight basis elements shown in Eq. (9).

The vector space defined by the canonical basis elements has dimension 8. It is therefore not the same as the vector space of the usual vectors of real space, which have three dimensions, or the complex vectors of the usual spinor representation of quantum states, which have 2 complex dimensions. The vector space of operators is simply a convenient abstraction.

Knowing the multiplication rules for $\hat{x}$, $\hat{y}$, and $\hat{z}$, we can derive the multiplication rule for more general spin operators. Given a vector $\vec{k}$, a natural Pauli algebra element to associate with it is $\hat{k}$ defined as:

$$\begin{aligned} \hat{k} &= \vec{k} \cdot (\hat{x}, \hat{y}, \hat{z}), \\ &= k_x \hat{x} + k_y \hat{y} + k_z \hat{z}. \end{aligned} \tag{10}$$

Let $\vec{u}$ and $\vec{v}$ be unit vectors. One finds that

$$\begin{aligned} \hat{u}\,\hat{v} &= (\vec{u} \cdot (\hat{x}, \hat{y}, \hat{z}))(\vec{v} \cdot (\hat{x}, \hat{y}, \hat{z})), \\ &= (u_x \hat{x} + u_y \hat{y} + u_z \hat{z})(v_x \hat{x} + v_y \hat{y} + v_z \hat{z}), \\ &= \vec{u} \cdot \vec{v} + \hat{x}\hat{y}\hat{z}\, (\vec{u} \times \vec{v}) \cdot (\hat{x}, \hat{y}, \hat{z}), \\ &= \vec{u} \cdot \vec{v} + \hat{x}\hat{y}\hat{z}\, \widehat{(\vec{u} \times \vec{v})}. \end{aligned} \tag{11}$$

In the last line of the above, $\vec{u} \cdot \vec{v}$ is a scalar, while the rest of the line is the constant $\hat{x}\hat{y}\hat{z}$ multiplied by the Pauli algebraic element associated with the vector $\vec{u} \times \vec{v}$.

This sort of construction, where a scalar is added to something that looks sort of like a vector, is common to the Pauli algebra, and Clifford algebras in general, but it can seem rather unsettling to those who are accustomed to separating physical objects according to their tensor properties. Anything written in the Pauli algebra can also be written in tensors by separating the algebra according to its transformation properties. If one were to do this, one could loosen the restrictions of the algebra. The Pauli algebra notation (when generalized to a Clifford algebra) is sufficient for the standard model particles and since it is more restrictive than tensors, it is preferable.

It is also possible to convert Pauli algebra notation into a general Lie algebra or symmetry notation. This would also loosen the restrictions of the Pauli (or Clifford) algebra. The standard model is usually written with Lie algebra notation and this has given it great generality along with too many experimentally determined parameters. In

writing quantum mechanics using only Clifford algebra, we hope to reduce the number of these free parameters.

1.3. Pure Density States

In the usual formulation of quantum mechanics, the states are represented by eigenvectors of the operators. As noted above, the Pauli algebra can always be thought of as a vector space so we can always think of an element of it as a vector. But the basis set of the Pauli algebra is not uniquely defined so our representation in terms of vectors would depend on an arbitrary choice. This is inelegant.

Instead, we shall represent states by elements of the Pauli algebra itself without writing them as vectors. The term "eigenvector" is then inapplicable, so we will call them "eigenstates" instead. In the Pauli algebra operator formalism, the eigenstates are particularly simple to define. Let $\vec{u}$ be a unit vector and $\hat{u}$ be the Pauli operator for spin in the $\vec{u}$ direction as before. Then the quantum state that has spin $\pm 1/2$ in the $\vec{u}$ direction is represented by the operator:

$$\rho_{\pm u,a,b} = (a + b\,\hat{x}\hat{y}\hat{z})(1 \pm \hat{u}), \tag{12}$$

where a and b are arbitrary real constants. Note that the above satisfies the double sided eigenvalue equations:

$$\hat{u}\ \ \rho_{\pm u,a,b} = \rho_{\pm u,a,b}\ \ \hat{u} = \pm\rho_{\pm u,a,b}, \tag{13}$$

rather than only the single sided eigenvalue equations that spinors satisfy.

The arbitrary constants a and b can be eliminated by choosing a normalization. For the pure density operator states, the natural normalization requirement is to require that the states be idempotent. An idempotent is an element of an algebra that remains unchanged when squared:

$$\rho^2 = \rho. \tag{14}$$

Applying this to Eq. (12), we find that $a = 0.5$ and $b = 0$, giving a normalized eigenstate:

$$\rho_{\pm u} = 0.5(1 \pm \hat{u}), \tag{15}$$

which the reader may verify is idempotent.

All these idempotent pure states have a scalar part of 0.5. Consequently, if we sum any two of them, the result will not necessarily be 1, but it will have a scalar part of 1. For example, $\rho_{+x} + \rho_{+y} = 1 + (\hat{x} + \hat{y})/2$. And in particular, ρ_{+u} and ρ_{-u} sum to unity:

$$\rho_{+u} + \rho_{-u} = 1. \tag{16}$$

We say that ρ_{+u} and ρ_{-u} are complementary. Also, the product of ρ_{+u} and ρ_{-u} is zero:

$$\rho_{+u}\,\rho_{-u} = 0. \tag{17}$$

We say that the two of them form a set of "self annihilating idempotents".

Since $\hat{x}\hat{y}\hat{z}$ acts like the imaginary unit, we can rewrite the unnormalized quantum state given by Eq. (12) in "complex" form, relative to the normalized state:

$$\rho_{\pm u,r,\theta} = re^{i\theta\hat{x}\hat{y}\hat{z}}\ 0.5(1 \pm \hat{u}). \tag{18}$$

The above form allows quick computation of powers of the state. That is, $(\rho_{\pm u,r,\theta})^n = \rho_{\pm u,r^n,n\theta}$.

In addition to the normalized pure states, the Pauli operator algebra contains two other idempotents, 0 and 1. In mathematics, an idempotent is said to be "primitive" if it is not zero, and it cannot be written as the sum of two non zero idempotents. Since 1 can be written as a sum of two idempotents it is not primitive. However, the pure density states cannot be written as the sum of two idempotents and so are primitive.

As mentioned above, two complementary primitive idempotents will sum to unity and multiply to zero. Mathematicians call the summing to unity property "complete", and the multiplication to zero "self annihilating." Thus two complementary primitive Pauli idempotents form a "complete set of annihilating primitive idempotents." In more complicated Clifford algebras, a complete set of annihilating primitive idempotents will be larger. The size of such a set is always a power of two. A complete set of annihilating primitive idempotents of the Dirac algebra will have 4 elements. The four Dirac states could be chosen as the spin up and down electron, and the spin up and down positron. Alternatively one could choose to split the algebra into left and right handed states and this is more convenient for elementary particle work.

1.4. The Measurement Algebra

In addition to summing to unity, the difference between two complementary normalized pure density states is also particularly simple:

$$\rho_{+u} - \rho_{-u} = \hat{u}. \tag{19}$$

From $\hat{x}$, $\hat{y}$, and $\hat{z}$ we can generate the whole operator algebra. Since we can obtain any of these as differences of pure density states, for example, $\hat{x} = \rho_{+x} - \rho_{-x}$, we can generate the whole Pauli operator algebra from just the pure density states. This is particularly elegant as this formalism has not split its mathematical objects into states and operators, but instead has written everything in terms of the only objects that are absolutely necessary for a quantum theory, the states.

In addition to the pure states, the density operator formalism also allows the description of statistical mixtures. The pure states correspond to the quantum states after spin is measured, and consequently, following Julian Schwinger[1], we will call this the "measurement algebra".

Schwinger writes the elements of his measurement algebra in the form $m(a)$, where "a" is a particular outcome of a measurement. In the present case, a could stand for spin being measured as +1/2 in some given direction, $\vec{u}$. Our notation would represent the same thing as ρ_u.

The interpretation of the primitive idempotents as measurements rather than pure

density operator states is subtle and powerful. What Schwinger did was to equate the quantum object with the measurement that would filter that particular sort of quantum object out of a beam of more general objects. Instead of analyzing the quantum states (which we can only understand through experiment), Schwinger chose to analyze the experiment itself.

Our work in this paper has taken Schwinger's interpretation a further step. We have written his measurement algebra in terms of the spin operator algebra. But the spin operator algebra is written entirely in terms of geometric objects, so this gives us a geometric description of both the elementary particles and the field configuration of the Stern-Gerlach experiment that picks them out.

Each measurement corresponds to a Stern-Gerlach experiment (see Fig. (1) and Fig. (2)) that allows the passage only of particles with the given spin state. We will call these "Stern-Gerlach filters", where the Stern-Gerlach experiment, by contrast, has two outputs. And while we will stick to spin in this paper, we can imagine that more general Stern-Gerlach filters could be modeled that would pick out other quantum numbers, for example, charge.

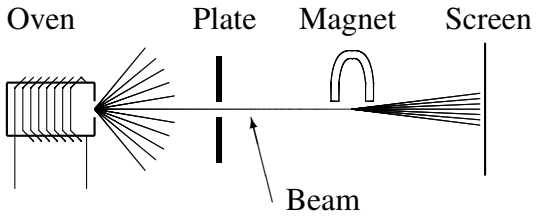

FIGURE 1. The Stern-Gerlach experiment: Atoms are heated to a gas in an oven. Escaping atoms are formed into a beam by a small hole in a plate. A magnet influences the beam, which then forms a figure on a screen.

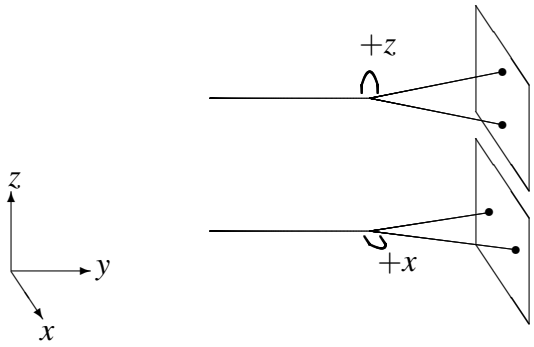

FIGURE 2. Two Stern-Gerlach experiments, one oriented in the $+z$ direction, the other oriented in the $+x$ direction. A Stern-Gerlach filter would let one of the two output beams go on, while the other is stopped.

If we are to think of the measurement algebra as an "algebra", we must define addition and multiplication. These are inherited from the definition of addition and multiplication in the operator algebra, but in the measurement algebra they receive new interpretations. These are again due to Julian Schwinger.

Given two Stern-Gerlach filters, we can hook them together two ways. We can arrange one to provide the input to the other so that particles must traverse both filters. We will

represent this by multiplication. The property of idempotency then models the physical fact that once particles are measured to have a property, a second measurement gives the same result. Consequently, two identical filters put in sequence are equivalent to a single filter; $\rho^2 = \rho$.

The other way we can arrange two filters is to place them side by side, so that a particle can pass either way. This we will represent by addition, for example, $\rho_a + \rho_b$. If we add two identical primitive idempotents, the result is twice a primitive idempotent, while the physical result is no different from a single filter. Thus we have to interpret real multiples of a filter as just a change to the normalization.

Since 1 is the multiplicative identity, that is, $1\rho = \rho$, we must interpret 1 as corresponding to a filter that makes no restriction, that is, a free beam. No particle can pass through two consecutive complementary filters, for example, the first oriented to pass only spin up, and the second oriented to pass only spin down. Looked at mathematically, the product of two complementary primitive idempotents is zero, so we interpret 0 as a filter that passes no particles, a beam stop.

1.5. Complex Multiples of Primitive Idempotents

We now consider a series of various filters that begin and end with the same filter, but have other filters in between. As a first example:

$$O_{xyx} = \rho_x \, \rho_y \, \rho_x. \tag{20}$$

Physically, such a compound filter will be equivalent to a lossy ρ_x filter. Computing:

$$\begin{aligned} O_{xyx} &= \rho_x \, \rho_y \, \rho_x, \\ &= 0.5(1+\hat{x})0.5(1+\hat{y})0.5(1+\hat{x}), \\ &= 0.125(1+2\hat{x}+\hat{y}+\hat{x}\hat{y}+\hat{y}\hat{x}+\hat{x}\hat{y}\hat{x}), \\ &= 0.125(2+2\hat{x}) = 0.5 \times 0.5(1+\hat{x}), \\ &= 0.5 \, \rho_x, \end{aligned} \tag{21}$$

we find that we obtain just half the primitive idempotent. The physical effect is that the beam amplitude is reduced by half. The energy of the beam, or the number of particles, is reduced to a quarter of that which would pass a simple ρ_x filter. This is what we expect. Just half the particles passing the first ρ_x will pass the ρ_y, and just half of these will pass the final ρ_x.

Now place a ρ_z after the ρ_y filter. There are now four filters in the sequence; we begin with an x, then a y, a z, and finally another x:

$$O_{xzyx} = \rho_x \, \rho_z \, \rho_y \, \rho_x. \tag{22}$$

Since there are now three perpendicular transitions, x to y, y to z, and z back to x, we expect that $2^{-3} = 0.125$ of the particles in the beam will survive, and so the amplitude will be equivalent to a $1/\sqrt{8}$ of the ρ_x. Computing as before, we find this to be true, but

we also find a complex phase:

$$\begin{aligned} O_{xzyx} &= \rho_x\,\rho_z\,\rho_y\,\rho_x, \\ &= 0.5(1+\hat{x})0.5(1+\hat{z})0.5(1+\hat{y})0.5(1+\hat{x}), \\ &= 0.0625(2+2\hat{x}-2\hat{y}\hat{z}-2\hat{x}\hat{y}\hat{z}), \\ &= 0.25(1-\hat{x}\hat{y}\hat{z})\,\rho_x = \sqrt{1/8}(1-\hat{x}\hat{y}\hat{z})/\sqrt{2}, \\ &= \sqrt{1/8}\,e^{-\hat{x}\hat{y}\hat{z}\,\pi/4}\,\rho_x. \end{aligned} \tag{23}$$

Note that there are three transitions, x to y, y to z, and finally z to x. We might expect that the above can be put into a cube form.

The $(1+\cos(\theta))/2$ rule gives probabilities of $1/2$, and therefore amplitudes of $1/\sqrt{2}$ for each transition. Thus the $1/\sqrt{8}$ in the above can be written instead as $(1/\sqrt{2})^3$. To write the $\exp(-\hat{x}\hat{y}\hat{z}\pi/4)$ as a cube, we change the $/4$ to $/12$. This puts O_{xzyx} in form:

$$\begin{aligned} O_{xzyx} &= (\exp(-\hat{x}\hat{y}\hat{z}\pi/12)/\sqrt{2})^3, \\ &= (\exp(-i\pi/12)/\sqrt{2})^3. \end{aligned} \tag{24}$$

Since $\hat{x}\hat{y}\hat{z}$ acts as an imaginary unit in the Pauli algebra, we have replaced it in the last line of the above with i, to save space.

The complex phase $\exp(i\pi/12)$ will reappear again and again in this paper. To save space, we abbreviate this as ζ:

$$\zeta = \exp(+i\pi/12), \quad \zeta^* = \exp(-i\pi/12). \tag{25}$$

Then we have $O_{xzyx} = (\zeta^*/\sqrt{2})^3$.

More generally, let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three unit vectors, and consider the composite filter consisting of:

$$O_{wvu} = \rho_w\,\rho_v\,\rho_u. \tag{26}$$

This must be a complex multiple of $\rho_w\,\rho_u$. Denote the complex multiple as k_{wvu}:

$$\rho_w\,\rho_v\,\rho_u = k_{wvu}\,(\rho_w\,\rho_u). \tag{27}$$

It is possible to show that k_{wvu} is given by:

$$k_{wvu} = \sqrt{r_{wvu}}\,e^{-\hat{x}\hat{y}\hat{z}\,s_{wvu}/2}, \tag{28}$$

where r_{wvu} is a real number giving the reduction in amplitude by the familiar $0.5(1+\cos(\theta))$ formula through the two transitions:

$$r_{wvu} = 0.5(1+\vec{u}\cdot\vec{v})\;\;0.5(1+\vec{v}\cdot\vec{w})/0.5(1+\vec{u}\cdot\vec{w}), \tag{29}$$

and s_{wvu} is a real number giving the oriented area of the spherical triangle with vertices given by $\vec{u}$, $\vec{v}$, and $\vec{w}$. See Fig. (3) for a hint on how to prove this.

The above formula for reduction will reduce "almost every" product of primitive idempotents, but it will have difficulties when the first and last primitive idempotent are complementary. In such cases, Eq. (29) will have a division by zero. This can also

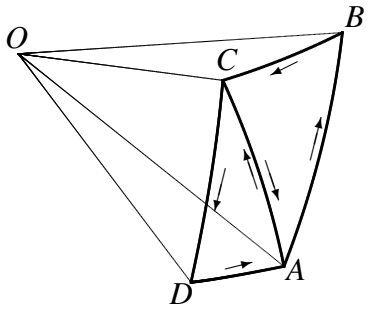

FIGURE 3. Two adjoining spherical triangles on the unit sphere. S_{uvw} is additive, that is, $S_{ABCD} = S_{ABC} + S_{ABD}$, and therefore the complex phase of a series of projection operators can be determined by the spherical area they encompass.

happen in an intermediate calculation. In addition, the reduction will always leave the first and last primitive idempotents unreduced.

2. SPINORS

In any standard introduction to quantum mechanics density matrices are derived from spinors or state vectors, and this makes it appear that density matrices are a less fundamental object than the spinors. In the previous section we showed not only that the density matrix formalism stands on its own, we showed that the density operator formalism is naturally defined in a geometric language.

In this section, we continue the reversal of roles between density matrices and spinors by showing that spinors can be naturally developed and defined entirely in the density operator formalism. By seeing spinors from the density operator point of view, we will gain insight into some of the odd behavior of spinors.

2.1. Bras and Kets from the Vacuum

The conversion from a spinor to a density matrix is well known. One multiplies the ket by the bra:

$$\rho_x = |x\rangle\langle x|. \tag{30}$$

The normalization of a density matrix is unique, as we saw in Eq. (15), a normalized density operator has 0.5 as its scalar part. In contrast, a spinor cannot be uniquely normalized, but is always normalized subject to an arbitrary phase, or gauge. For θ a real number, one can transform the bra and ket as follows:

$$\begin{array}{rcl} |x\rangle & \rightarrow & e^{+i\theta}\,|x\rangle, \\ \langle x| & \rightarrow & e^{-i\theta}\,\langle x|, \end{array} \tag{31}$$

to obtain a new bra and a new ket that represent the same quantum state. In multiplying these two together to give ρ_x, the phases cancel, so these new spinors still produce the same density operator.

Thus the spinor formalism includes arbitrary, non physical, information that is eliminated when one converts from the spinor to the operator formalism. The density operator formalism includes all the information of the quantum state, the extra phase information included in the spinor formalism is unphysical. To reverse the procedure, to convert from the operator formalism to spinor, we must reintroduce arbitrary phase information.

To define spinors in the density operator formalism, we will choose an arbitrary pure density operator. This arbitrary choice will provide the necessary arbitrariness of a spinor representation. Following Julian Schwinger [1], we will call it the "vacuum"[1] and label it as ρ_0. Given a density operator ρ_χ, we define the corresponding spinors as:

$$\begin{array}{rcl} |\chi\rangle & \equiv & \rho_\chi\, \rho_0, \\ \langle\chi| & \equiv & \rho_0\, \rho_\chi. \end{array} \tag{32}$$

The above defines the spinors entirely within the operator language. We now show that these definitions lead to the correct behavior of spinors.

First, note that our spinors can be multiplied to give back the density operator ρ_χ:

$$\begin{array}{rcl} |\chi\rangle\langle\chi| & = & \rho_\chi\, \rho_0\, \rho_0\, \rho_\chi, \\ & = & \rho_\chi\, \rho_0\, \rho_\chi, \\ & = & k_{\chi 0 \chi}\, \rho_\chi, \end{array} \tag{33}$$

where $k_{\chi 0 \chi}$ is the complex number defined in Eq. (27). Note that there will be a division by zero if ρ_0 is antiparallel to ρ_χ, but since this is only a single direction, the definition is correct almost everywhere. From this we see that our spinors are not normalized, but otherwise do multiply to give back the density operator they were made from. The normalization can be obtained by multiplying the spinors by a scale factor and an arbitrary choice of phase: $|k_{\chi 0 \chi}|^{-0.5} \exp(i\theta)$.

Second, we need to be able to use our bras and kets to compute matrix elements. That is, for any operator M, we need to be able to compute $\langle\chi|\, M\, |\chi\rangle$:

$$\langle\chi|\, M\, |\chi\rangle = \rho_0\, \rho_\chi\, M\, \rho_\chi\, \rho_0. \tag{34}$$

The above is a product of operators that begin and end with the same spin operator, ρ_0. As such, it must be a complex multiple of ρ_0. To put this into physical terms, a complicated filter that begins and ends with the same Stern-Gerlach filter must act as a single Stern-Gerlach filter of that same type, but multiplied by a complex constant.

Thus the matrix elements we get from our operator spinors will not be complex numbers, but instead will be complex multiples of the vacuum operator ρ_0. In Eq. (34), the complex multiple could be written $k_{0\chi M\chi 0}$. We need to show that complex multiples

[1] Schwinger calls this object the "vacuum", and the bras and kets made from it are respectively called annihilation and creation operators, as their algebra with each other and the vacuum gives the usual vacuum relations. He was looking at a formalism for quantum field theory.[1]

of the vacuum operator act like complex numbers. Let a and b be two complex numbers, then:

$$\begin{aligned}(a\,\rho_0)+(b\,\rho_0) &= (a+b)\,\rho_0,\\ (a\,\rho_0)(b\,\rho_0) &= (ab)\,\rho_0\,\rho_0=(ab)\,\rho_0,\end{aligned} \tag{35}$$

so the conversion from complex numbers to complex multiples of ρ_0 is consistent with complex addition and multiplication. We leave the remaining details, such as correcting the normalization, to the reader.

By defining spinors in terms of density operators, we now have several more or less independent choices to make when choosing how to represent quantum states. We can use spinors or operators. That is, our objects can be either linear or bilinear. In representing these objects, we can use spinors as the fundamental object or we can use operators as the fundamental object.

We can assume our fundamental quantum states to be either density operators or spinors. If we choose operators, we have the further choice of using the Pauli spin matrices to represent them, or we can use the geometric language and leave them in $\hat{x}$, $\hat{y}$, and $\hat{z}$. These are very obvious choices. If we choose our fundamental quantum states to be represented by spinors, the obvious choice is to follow the usual methods of quantum mechanics. However, it is also possible to define spinors in a geometric representation and this has been done before, most notably by David Hestenes who geometrized both the Pauli algebra with R. Gurtler, [2] and the Dirac algebra.[3]

The methods used here to geometrize the density operators of the Pauli algebra work analogously in the Dirac algebra. That is, any 4x4 matrix A can be written as a sum over complex multiples of products of gamma matrices:

$$A = a\hat{1} + a_0\gamma_0 + a_1\gamma_1 ... + a_{0123}\gamma_0\gamma_1\gamma_2\gamma_3, \tag{36}$$

where $\hat{1}$ is the 4x4 unit matrix, and a_χ are sixteen complex constants. Thus any such operator can be immediately geometrized by replacing the gamma matrices γ_0, γ_1, γ_2, and γ_3, with, respectively, $\hat{t}$, $\hat{x}$, $\hat{y}$, and $\hat{z}$:

$$A \equiv a + a_0\hat{t} + a_x\hat{x} + ... + a_{0123}\hat{t}\hat{x}\hat{y}\hat{z}. \tag{37}$$

In the above, the operators satisfy the usual anticommutation relations of the gamma matrices, and square to whichever signature the user prefers.

This method of geometrizing the operators is very obvious and simple. The case with defining spinors geometrically, without using density operators and a vacuum state, is not so obvious, not at all unique–especially for the Dirac spinors, and different authors have chosen different methods.[4] The common problem in these approaches is that the operators, which are naturally geometrized, have more degrees of freedom than spinors. So when one geometrizes the spinors one must choose a method of getting rid of the extra degrees of freedom. The obvious way of reducing degrees of freedom in an algebra is to multiply the elements by a projection operator. In the density operator formalism given here, the pure states are projection operators, and thus the vacuum state is a projection operator and acts to reduce the number of degrees of freedom in the

algebra.[2]

For representations of spin$-1/2$ particles as density matrices, we now have three choices. For a pure state giving spin $+1/2$ in the $+x$ direction, the choices are:

$$\begin{array}{rrcl} \text{D.M. from Pauli Spinor:} & \sqrt{0.5}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sqrt{0.5}\begin{pmatrix} 1 & 1 \end{pmatrix} & = & 0.5\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \text{D.M. as Pauli Operator:} & 0.5(1+\sigma_x) & = & 0.5\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \text{Geometric Operator:} & 0.5(1+\hat{x}) & = & 0.5(1+\hat{x}). \end{array} \tag{38}$$

Of the above, the third row, the operator representation of the density operator, requires no choice of representation, is unique, simple, and the notation is the shortest. The traditional choice is the top row. The traditional method begins with a ket $(1,1)/\sqrt{2}$ that is obtained by solving a eigenvalue equation. This algebra is unneeded in the second and third rows.

For representations of spin$-1/2$ particles as spinors, we have discussed two choices, the traditional and a method using a vacuum state. To illustrate the operator method, we need to choose a vacuum state. We will use $\rho_0 = \rho_z = 0.5(1+\hat{z})$. The two methods are:

$$\begin{array}{rl} \text{Pauli Spinor:} & \sqrt{0.5}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \text{Operator and Vacuum:} & \sqrt{0.125}(1+\hat{x})(1+\hat{z}), \end{array} \tag{39}$$

where the $\sqrt{0.5}$ in the second row comes from choosing a normalization to give $\langle x|x\rangle = \rho_0 = 0.5(1+\hat{z})$. The vacuum method is as messy as the traditional method, which is why it has not gained in popularity. Between the density operator and spinor representations combined, we have discussed five different methods of representing a quantum spin$-1/2$ state. Calculations in these five methods are compared in Section (3).

2.2. Naughty Spinor Behavior

Any Pauli pure state density operator has a spin orientation. This gives us a geometric interpretation of the source of the arbitrary complex phase seen in the spinor formalism. In this section we look back at the spinor formalism from the perspective of the density operator formalism.

When a spinor is rotated by 2π it does not return to its original value but instead is multiplied by -1. Since phases are arbitrary even in normalized spinors, this has no

[2] In [2] and most of his later work, Hestenes defines spinors as the even subalgebra of the operators, for example $a_1 + a_{xy}\hat{x}\hat{y}$. The method used in this paper is equivalent to the mathematically more natural, "ideal" method that Hestenes used in an earlier work.[5]

physical consequences.[3] Since density operators do not have arbitrary complex phases, this unphysical behavior on rotation does not occur. On the other hand, running a particle through a series of spin measurements with various orientations is physical, and in the density operator formalism does result in a phase change, as was shown in Subsection (1.5).

The operator that rotates a spinor by an angle λ around a rotation axis defined by the vector $\vec{u}$ is simply:

$$U(\lambda) = e^{i\lambda\hat{u}/2} \tag{40}$$

Let $|v\rangle$ be an arbitrary ket. We can write:

$$|v\rangle = 0.5(1+\hat{u})\,|v\rangle + 0.5(1-\hat{u})\,|v\rangle. \tag{41}$$

Applying the rotation operator to this gives:

$$\begin{aligned} U(\lambda)\,|v\rangle &= U(\lambda)(1+\hat{u})/2\,|v\rangle + U(\lambda)(1-\hat{u})/2\,|v\rangle \\ &= 0.5e^{+i\lambda/2}(1+\hat{u})\,|v\rangle + 0.5e^{-i\lambda/2}(1-\hat{u})\,|v\rangle, \end{aligned} \tag{42}$$

where we have taken advantage of the fact that $(1+\hat{u})/2$ is an eigenstate of $\hat{u}$.

Putting $\lambda = 2\pi$ gives

$$\begin{aligned} U(2\pi)|v\rangle &= -0.5(1+\hat{u})\,|v\rangle - 0.5(1-\hat{u})\,|v\rangle, \\ &= -|v\rangle. \end{aligned} \tag{43}$$

To rotate a bra, one puts the rotation operator on the other side, and because of the complex conjugate, the spin operator takes a negative angle, $U(-\lambda)$

For a density operator, or any other operator, the rotation operator must be applied to both sides of the density operator. This gives two factors of -1. Thus a density operator is unmodified when rotated through 2π using the rotation operators.

Applying the rotation operator to a spinor made from density operators, we see the source of the factor of -1. When a spinor made from density operators is to be rotated by spinors, one must include an extra rotation operator to rotate the vacuum state. In the density operator formalism we have:

$$\begin{aligned} |v\rangle &= |v\rangle\langle v||0\rangle\langle 0|, \\ |v\rangle' &= U(\lambda)|v\rangle\langle v|U(-\lambda)\,U(\lambda)|0\rangle\langle 0|U(-\lambda) \\ &= U(\lambda)|v\rangle\langle v|0\rangle\langle 0|U(-\lambda). \end{aligned} \tag{44}$$

Putting $\lambda = 2\pi$ leaves the state unchanged, consistent with the fact that density operators are unchanged by rotations of 2π. Thus, from the density operator point of view, the -1 that a ket takes on rotation by 2π is a consequence of failing to rotate the vacuum operator.

In this paper we consider density operators at a single point in space, for example

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[3] That is, the density matrix contains all physically relevant information but in it the arbitrary phase is not present. Phases are only significant when different phases are compared.

$\rho(x)$. These are not sufficient for conditions where a wave function is defined over a spatial region. In that case wave functions take two points in space, $\rho(x,x')$. These are defined by multiplying two wave functions:

$$\rho(x,x') = \psi^*(x)\psi(x') \tag{45}$$

Given a wave function ψ, one can always multiply the wave function by an arbitrary complex phase, $\psi' = \exp(i\theta)\psi$. Doing this leaves $\rho(x,x')$ unchanged, so in this case the complex phase information has also been eliminated. Since x and x' can be two different points in space, the relative phase between two points is still present in $\rho(x,x')$. As with the situation discussed in this paper, $\rho(x,x')$ contains all the physical information present in $\psi(x)$, but has lost an arbitrary phase.

As in the spinor case, one can obtain a wave function $\psi(x)$ from a density operator $\rho(x,x')$. To do this, one simply assigns $\psi(x) = \exp(i\theta)\ \rho(0,x)$ where "0" is an arbitrarily chosen position in space (or spacetime) and θ is an arbitrary real number. The two descriptions of a quantum state, $\psi(x)$ and $\rho(x,x')$ are equivalent, but a particular advantage of the usual $\psi(x)$ description is that of linear superposition.

2.3. Linear Superposition

An advantage of spinors over density operators is that they allow linear superposition. On the other hand, the usual spinor formalism is not linear in the physical sense that Electricity and Magnetism is linear. When one triples a given spinor, the result is a spinor representing the same state, not a spinor for a state with three times as many particles or particles with charges that are three times stronger. In this subsection we explore the meaning of linear superposition in spinors from the point of view of the density operator formalism.

Given two spinors $|u\rangle$ and $|v\rangle$, and two complex numbers, a and b, we might define their linear superposition as:

$$|a\hat{u} + b\hat{v}\rangle = a\,|u\rangle + b\,|v\rangle. \tag{46}$$

Unfortunately, the above equation is nonsense unless one also specifies a way of choosing the arbitrary complex phases of $|u\rangle$ and $|v\rangle$. For example, the sum $|+z\rangle + |-z\rangle$ can be variously computed as:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{i\theta} \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}, \tag{47}$$

which, depending on the value of θ, can be any spinor oriented in the $x-y$ plane.

Linear superposition is not well defined for quantum states–it is defined only for spinor representations of quantum states where one makes a choice of their relative phases . This suggests that linear superposition is a convenience of the mathematics rather than a deep attribute of physics.

In the operator representation of spinors, the choice of vacuum defines all the complex

phases of the spinors, so with such a choice of vacuum, the ket $|a\hat{u}+b\hat{v}\rangle$ becomes well defined:

$$\begin{aligned} |\hat{u}\rangle &= \rho_u\,\rho_0, \\ |\hat{v}\rangle &= \rho_v\,\rho_0, \\ |a\hat{u}+b\hat{v}\rangle &= (a\rho_u+b\rho_v)\rho_0. \end{aligned} \tag{48}$$

When one is considering linear superpositions of spinors, one must choose a phase for each spinor. In the density operator case, the choice of vacuum state defines a consistent set of phases for all the spinors (other than the state that are zeroed by the vacuum state). Bras are also well defined

$$\langle a\hat{u}+b\hat{v}|a = \rho_0\,(a^*\rho_u+b^*\rho_v), \tag{49}$$

except that they take complex conjugates for the coefficients.

Any non zero complex 2-vector can be interpreted as a (not necessarily normalized) spinor. When one makes a density matrix out of such a spinor, one can only obtain density matrices corresponding to pure states. Thus linear superposition in spinors, when translated into density operators, also manages to avoid statistical mixtures, which we now show.

Let ρ_0 be the vacuum, i.e. an arbitrary density operator, and let V and W be arbitrary operators. We will prove that the product $V\,\rho_0\,W$ is a complex multiple (possibly zero) of a primitive idempotent. This includes the case where $V = a\rho_u + b\rho_v, W = 1$ and $V = 1, W = a^*\rho_u + b^*\rho_v$, so by proving this we will prove that the linear superposition defined above for density operators does result in spinors that can be added together to give a complex multiple of a pure state.

Since ρ_0 is a primitive idempotent, any product of operators that begin and end with ρ_0 will be a complex multiple of ρ_0. In particular, $\rho_0\,W\,V\,\rho_0$ is such a product and gives a complex multiple of ρ_0:

$$\rho_0\,W\,V\,\rho_0 = k_{0WV0}\,\rho_0 \tag{50}$$

where k_{0WV0} is a complex number. Then the square of $V\,\rho_0\,W$ is a complex multiple of $V\,\rho_0\,W$:

$$\begin{aligned} (V\,\rho_0\,W)^2 &= V\,(\rho_0\,W\,V\,\rho_0)\,W, \\ &= V\,(k_{0WV0}\,\rho_0)\,W, \\ &= k_{0WV0}\,(V\,\rho_0\,W). \end{aligned} \tag{51}$$

Thus if k_{0WV0} is non zero, $V\,\rho_0\,W/k_{0WV0}$ is idempotent. It remains to show that $V\,\rho_0\,W/k_{0WV0}$ is primitive. This can be done by computing the trace and showing that it is unity, an exercise we leave to the reader.

2.4. Statistical Mixtures

Suppose we use a Stern-Gerlach apparatus, oriented in the u direction, to split a beam of unpolarized spin$-1/2$ particles into two beams. One beam will have particles that can be represented by $\rho_{+u} = 0.5(1+\hat{u})$ and will have spin up, the other beam will have

particles represented by $\rho_{-u} = 0.5(1 - \hat{u})$. These two elements add to unity:

$$1 = \rho_{+u} + \rho_{-u}. \tag{52}$$

Since these two beams have been split from one unpolarized beam, we will represent an unpolarized spin$-1/2$ particle with the density matrix 1:

$$\rho = 1 \rightarrow \rho_{+u} + \rho_{-u}. \tag{53}$$

The reader should note that this choice of normalization, which follows Julian Schwinger[1], is twice that chosen in the usual density matrix theory, which arranges for the trace to be unity:

$$\rho_{\text{usual}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}. \tag{54}$$

Normalizing so that the trace is unity makes sense because the trace is so important in the usual density matrix theory. The pure states also have trace unity, so this is convenient for a theory that concentrates on the attributes of single particles. Rather than the trace, we will normalize with the "scalar[]" function. This picks out the scalar part of a geometric object. In the Pauli algebra, this is twice the trace. In the Dirac algebra, it is four times the trace.

The primary advantage to using the scalar function to normalize is that in a geometric theory, the scalar function is a more natural function than the trace. A secondary advantage is that we unite the representation of a Stern-Gerlach experiment with the particles that it selects out. For the Pauli algebra this is of little importance, but it becomes convenient in more complicated algebras.

The heart of the classic Stern-Gerlach filter is a non-uniform magnetic field. Particles with different spin react differently to this magnetic field. This is described as the interaction between spin and the non-uniform magnetic field. If the magnetic field is non-uniform in the z direction, then the particles will be separated according to their spin in the z direction. In this sense, there is an intimate relationship between the geometry of the magnetic field and the geometry of the selected particles.

For particles described by the Pauli algebra, and ignoring spatial wave functions, the classic Stern-Gerlach experiment is the only measurement we can make on them. That is, we can measure their spin in some direction, that is all. We can imagine generalizations of the Stern-Gerlach experiment that split beams of more complicated particles. It is in these more complicated Stern-Gerlach experiments that we find another motivation for using a scalar normalization.

For particles described by the Dirac algebra, we could imagine a generalized Stern-Gerlach experiment that split an "unpolarized" beam into four parts, for example, spin-up electrons, spin-down electrons, spin-up positrons, and spin-down positrons. This would be a more complicated field configuration than the usual Stern-Gerlach experiment, but we could also send the unpolarized beam into a simple Stern-Gerlach experiment.

If we use scalar normalization, the mathematical objects we use to represent the various Stern-Gerlach experiments will be the same mathematical objects that we use to

represent the particles that are separated out by that particular Stern-Gerlach experiment. If the intensity of this beam is I, then the intensity after going through a classical Stern-Gerlach filter will be $I/2$, which is just I times the scalar part of the idempotent $(1+\hat{u})/2$ which we use to represent the Stern-Gerlach experiment. Similarly, a more general Stern-Gerlach experiment will separate out $1/4$ of the beam just as the scalar part of the Dirac algebra primitive idempotent is $1/4$.

In addition to splitting an unpolarized beam into two beams of pure state particles, we can also consider the opposite procedure, the uniting of two pure beams into a single beam. If the orientations of these two beams are opposite, they will combine into an unpolarized beam:

$$\rho_{+u} + \rho_{-u} = 1 = \rho. \tag{55}$$

In combining two beams like this, note that the orientation of the incoming beams does not appear in the output.

The expectation value of an operator M on a particle taken from a beam represented by a density matrix is given by the trace of the product. For the case of an unpolarized beam, $\rho = 1$, the expectation value is:

$$\langle M \rangle = \text{tr}(M\rho) = \text{tr}(M1) = \text{tr}(M), \tag{56}$$

and thus the expectation value of the operator does not depend on the orientation of the two beams that were combined. For all practical purposes, the orientations of the beams that united to produce the unpolarized beam has been lost.

If we suppose that the spinor representation is fundamental, it would be natural to suppose that a single particle taken from the no longer polarized beam does have an orientation, but the information simply cannot be extracted from the particle by experiment. The rules of quantum mechanics hide this information from us. For this reason, physicists who treat the spinor as fundamental will tend to treat a statistical mixture as only valid for an ensemble of quantum states. In the density matrix formalism this is unnecessary. All the particles in mixed beam are represented by the same mathematical object with no hidden information to be mixed up by an ensemble.

3. EXAMPLE CALCULATIONS

This section illustrates the five methods of describing quantum states defined above by performing the same calculation using each. We compute the transition probability from a state with spin $+1/2$ in the $\vec{z} = (0,0,1)$ direction to a state with spin $+1/2$ in the $\vec{u} = (0, \sin(\theta), \cos(\theta))$ direction. These are two directions with an angle of θ between them. The expected result is $0.5(1+\cos(\theta))$.

3.1. Pauli Spinor

For the traditional Pauli spinor method, we first define the operators for spin in the $\vec{z}$ and $\vec{u}$ directions:

$$\begin{array}{rcl} \sigma_z & = & \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_u & = u_y\sigma_y + u_z\sigma_z = & \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ i\sin(\theta) & -\cos(\theta) \end{pmatrix}. \end{array} \tag{57}$$

One next finds eigenvectors for the above. If $\chi^2 = 1$, then a convenient trick is that since $(1+\hat{\chi})$ is an eigenmatrix (or eigenstate) of the operator $\hat{\chi}$, one can take any non zero column of $(1+\hat{\chi})$ as our eigenvector. The $(1+\hat{\chi})$ matrices are:

$$\begin{array}{rcl} 1+\sigma_z & = & \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \\ 1+\sigma_u & = & \begin{pmatrix} 1+\cos(\theta) & -i\sin(\theta) \\ i\sin(\theta) & 1-\cos(\theta) \end{pmatrix}. \end{array} \tag{58}$$

For $1+\sigma_z$ there is only one nonzero column, but $1+\sigma_u$ gives us two choices. We will use the leftmost column. Our eigenvectors are:

$$|z\rangle = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad |u\rangle = \begin{pmatrix} 1+\cos(\theta) \\ i\sin(\theta) \end{pmatrix}. \tag{59}$$

From the kets, one finds the bras:

$$\begin{array}{rcl} \langle z| & = & \begin{pmatrix} 2 & 0 \end{pmatrix}, \\ \langle u| & = & \begin{pmatrix} 1+\cos(\theta) & -i\sin(\theta) \end{pmatrix}. \end{array} \tag{60}$$

The transition probability is given by:

$$\begin{array}{rcl} P(z\rightarrow u) & = & \langle z|u\rangle\langle u|z\rangle / (\langle z|z\rangle\langle u|u\rangle), \\ & = & 2(1+\cos(\theta))2(1+\cos(\theta))/4\ ((1+\cos(\theta))^2+\sin^2(\theta)), \\ & = & (1+\cos(\theta))^2/(1+2\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)), \\ & = & 0.5(1+\cos(\theta)), \end{array} \tag{61}$$

as expected. If one were to assign this problem to an undergraduate student in quantum mechanics, one would expect a solution along this line, perhaps with the alternative of first normalizing the bras and kets.

3.2. Density Matrix from Pauli Spinor

The second commonly taught calculational method is with density matrices defined inside the Pauli algebra. In this method, one begins with the bras and kets given by

Eq. (59) and Eq. (60), and converts them into density matrices:

$$\begin{aligned} |z\rangle\langle z| &= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \\ |u\rangle\langle u| &= \begin{pmatrix} 1+2\cos(\theta)+\cos^2(\theta) & -i\sin(\theta)(1+\cos(\theta)) \\ i\sin(\theta)(1+\cos(\theta)) & \sin^2(\theta) \end{pmatrix}. \end{aligned} \tag{62}$$

We normalize the above by dividing by their traces, which, after some algebra, gives the following matrices:

$$\begin{aligned} \rho_{\sigma z} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \rho_{\sigma u} &= 0.5\begin{pmatrix} 1+\cos(\theta) & -i\sin(\theta) \\ i\sin(\theta) & (1-\cos(\theta)) \end{pmatrix}. \end{aligned} \tag{63}$$

The transition probability is obtained by taking the trace of the product of these two matrices:

$$\begin{aligned} P(z\rightarrow u) &= \mathrm{tr}(\rho_{\sigma z}\,\rho_{\sigma_u}), \\ &= \mathrm{tr}\begin{pmatrix} 0.5(1+\cos(\theta)) & 0 \\ 0.5i\sin(\theta) & 0 \end{pmatrix}, \\ &= 0.5(1+\cos(\theta)). \end{aligned} \tag{64}$$

This method is more messy than the traditional. One would be a little surprised to see a student hand this in.

3.3. Density Matrix as Pauli Operator

Our next method of calculation also uses the Pauli spin matrices and the density operator form, but instead of finding the density matrices from bras and kets, we instead stay in the operator language. The eigenstate of the operator $\hat{u}$ is simply $1+\hat{u}$. One normalizes this to give a trace of unity by multiplying by 0.5:

$$\begin{aligned} \rho_{\sigma z} &= 0.5(1+\sigma_z) &&= 0.5\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \\ \rho_{\sigma u} &= 0.5(1+\sigma_u) &&= 0.5\begin{pmatrix} 1+\cos(\theta) & -\sin(\theta) \\ +i\sin(\theta) & 1-\cos(\theta) \end{pmatrix}. \end{aligned} \tag{65}$$

From here, the answer is obtained as in Eq. (64). It is interesting to note that this method requires little effort to normalize the states. In particular, no square roots appear form normalization. Nevertheless, an instructor might be surprised to see a student turn this calculation in.

3.4. Geometric Operator

The method of calculation with geometric operators is similar to that in the previous subsection, except there is no need to choose a representation for the algebra, so there are no matrices. This geometric operator method is by far the simplest. To illustrate this, let us make the problem harder by replacing the simple unit vectors $\vec{z}$ and $\vec{u}$, half of whose entries are zero, with two arbitrary unit vectors $\vec{v}$ and $\vec{w}$. The normalized density operators are:

$$\begin{array}{rcl} \rho_v & = & 0.5(1+\hat{v}), \\ \rho_w & = & 0.5(1+\hat{w}). \end{array} \tag{66}$$

The transition probability is given by the trace of the product $\rho_v\,\rho_w$. Since the trace is twice the scalar part, we have:

$$\begin{array}{rcl} P(v\rightarrow w) & = & 2 \text{ scalar } [(\rho_v\,\rho_w)], \\ & = & 2 \text{ scalar } [0.25(1+\hat{v}+\hat{w}+\hat{v}\hat{w})], \\ & = & 0.5+0.5 \text{ scalar } [\hat{v}\hat{w}]. \end{array} \tag{67}$$

In the above, the vector terms $\hat{v}+\hat{w}$ have disappeared because they contain no scalar part. For the scalar part of $(\hat{v}\hat{w})$ we use Eq. (11) and find that the scalar part is just $\vec{v}\cdot\vec{w}$. This gives:

$$\begin{array}{rcl} P(v\rightarrow w) & = & 0.5+0.5\,\vec{v}\cdot\vec{w}, \\ & = & .5(1+\cos(\theta)), \end{array} \tag{68}$$

where θ is the angle between the unit vectors $\vec{v}$ and $\vec{w}$. This gives the general solution in just a few lines of algebra, but it would be a rare classroom that saw it.

3.5. Spinors from Operator and Vacuum

The fifth and last method of making the calculation is to use the operator formalism with a vacuum. For the vacuum, we need to choose a state that will not be antiparallel to $\vec{z}$ or $\vec{u}$. These are in the $y-z$ plane, so we will choose the vacuum as $\rho_0=\rho_x=0.5(1+\hat{x})$. The vacuum technique is messy, but is manageable if we apply the methods of Subsection (1.5).

The density operators for $\hat{z}$ and $\hat{u}$ are given by the usual $0.5(1+\hat{\chi})$. We have $\hat{u}=\cos(\theta)\hat{z}+\sin(\theta)\hat{x}$, so:

$$\begin{array}{rcl} \rho_z & = & 0.5(1+\hat{z}), \\ \rho_u & = & 0.5(1+\cos(\theta)\hat{z}+\sin(\theta)\hat{y}). \end{array} \tag{69}$$

The vacuum state is $\rho_0=\rho_x$, so the ket states are:

$$\begin{array}{rcl} |z\rangle & = & \rho_z\,\rho_x, \\ |u\rangle & = & \rho_u\,\rho_x. \end{array} \tag{70}$$

and the corresponding bra states are:

$$\begin{array}{rcl} \langle z| & = & \rho_x\,\rho_z, \\ \langle u| & = & \rho_x\,\rho_u. \end{array} \tag{71}$$

The transition probability is the ratio of the spinor products as in the usual spinor calculation, Eq. (61):

$$P(z \to u) = \langle z|u\rangle\langle u|z\rangle / (\langle z|z\rangle\langle u|u\rangle), \tag{72}$$

but the division "/" is to be interpreted as a division within the subalgebra defined by ρ_x. Computing the four components $\langle z|u\rangle$, $\langle u|z\rangle$, $\langle z|z\rangle$, $\langle u|u\rangle$, separately, we need to compute the following four products of primitive idempotents:

$$\begin{array}{rcl} \langle z|z\rangle & = & \rho_x\,\rho_z\,\rho_x, \\ \langle z|u\rangle & = & \rho_x\,\rho_z\,\rho_u\,\rho_x, \\ \langle u|z\rangle & = & \rho_x\,\rho_u\,\rho_z\,\rho_x, \\ \langle u|u\rangle & = & \rho_x\,\rho_u\,\rho_x. \end{array} \tag{73}$$

In the above, the first and last equation have been simplified by using $\rho_z\,\rho_z = \rho_z$. The products can be further reduced using the equations of Subsection (1.5).

The first and fourth products are easiest as they define spherical triangles that have zero area. Therefore the products are real multiples of ρ_x. The $\rho_x\,\rho_z\,\rho_x$ calculation is:

$$\begin{array}{rcl} \rho_x\,\rho_z\,\rho_x & = & \sqrt{0.5(1+\cos(\theta_{xz}))0.5(1+\cos(\theta_{zx}))}\;\rho_x, \\ & = & 0.5\rho_x, \end{array} \tag{74}$$

where $\theta_{xz} = \pi/2$ is the angle between $\vec{x}$ and $\vec{z}$. The $\rho_x\,\rho_u\,\rho_x$ calculation is similar.

The second and third products of Eq. (73) define spherical triangles that have nonzero area and consequently there is a complex phase involved. But the two triangles, i.e. *xzu* and *xuz* are the same triangle but with opposite orientation. Therefore the complex phases of these two products will be complementary, and since the transition probability depends on the product of these, their complex phases will cancel. The product will be a real number. Consequently, we need only compute magnitudes. And the magnitudes of the second and third products are equal:

$$\begin{array}{rcl} \rho_x\,\rho_z\,\rho_u\,\rho_x & = & \sqrt{0.5(1+\cos(\theta_{xz}))0.5(1+\cos(\theta_{zu}))0.5(1+\cos(\theta_{ux}))}\;\rho_x, \\ & = & \sqrt{0.125(1+\cos(\theta))}\;\rho_x, \end{array} \tag{75}$$

where the phase has been left off and we write θ for θ_{zu}.

In computing products of complex multiples of ρ_x, we need to keep track of the complex numbers only, the copies of ρ_x reduce to a single ρ_x because $\rho_x^2 = \rho_x$. The transition probability is thus given by the product of the magnitudes of the four $\langle\chi|\chi\rangle$

products:

$$\begin{aligned} P(z \rightarrow u) &= \langle z|u\rangle\langle u|z\rangle / (\langle z|z\rangle\langle u|u\rangle), \\ &= \sqrt{0.125(1+\cos(\theta))}\sqrt{0.125(1+\cos(\theta))}/(0.5)(0.5), \\ &= 0.5(1+\cos(\theta)). \end{aligned} \tag{76}$$

If the reader wishes to suffer a more complicated problem, he is invited to reproduce the calculation for the more general case, replacing $\vec{z}, \vec{u}$, and $\vec{x}$ with more arbitrary vectors.

4. BIGGER AND BETTER CLIFFORD ALGEBRAS

The Pauli algebra is used for representing non relativistic spin operators. Relativistic spin operator theory uses the Dirac algebra. Both these algebras are examples of Clifford algebras. Clifford algebras are defined by a set of "canonical basis vectors" that anticommute and square to ± 1. For the Pauli algebra, the canonical basis vectors are σ_x, σ_y, and σ_z and they all square to $+1$. The Dirac algebra adds one further vector, for time, but renames them γ_1, γ_2, γ_3, and γ_0. Two signatures are possible, with either the spatial or time basis vectors squaring to $+1$ and the rest squaring to -1, or the opposite. We will assume that the spatial basis vectors square to $+1$.

The extra canonical basis vector in the Dirac algebra over the Pauli algebra arises from the addition of 1 time dimension to the 3 of spacetime. A related fact is that the Dirac spinors are 4×1 vectors while the Pauli spinors are only 2×1. The increase in complexity has come about because of an increase in the number of dimensions assumed for the manifold on which the objects are defined.

Various string theories assume various numbers of hidden dimensions and in these theories the spinors become yet larger. The relationship between the size of the spinors and the number of hidden dimensions is a little subtle. If the algebra is defined to be the complex vector space over the products of canonical basis vectors (such as is the natural case for the Dirac algebra), then the size of the spinors doubles for every other dimension added. If the algebra is defined to be the real vector space over the products of canonical basis vectors (such as is the natural case for the Pauli algebra), then the spinors double in size or do not according to a complicated formula that depends on the signatures of the dimensions involved. The general solution for this case involves the Radon-Hurwitz numbers, which are available conveniently on the web at [6]. The complex case has no dependence on signature.

All these cases, the Pauli algebra, the Dirac algebra, and the more complicated Clifford algebras used in string theories, produce spinors that can be used to form density matrices. And the geometrization technique used here will give a natural geometrization of the density operators of those particles.

Suppose that we have a Clifford algebra and it is used to represent elementary particles in spinor form. A complete set of N particles, $\{|n\rangle, n = 1,2,...N\}$, means a set of states that are normalized and orthogonal, $\langle n|m\rangle = \delta_n^m$, and are complete in that any state can be written in terms of them. The set being complete means that $\sum_n \rho_n = 1$. Since the spinors are "normalized", we have that $\rho_n^2 = 1$, that is, it means that the corresponding density matrices are idempotent. "Orthogonal" means that $\rho_n \rho_m = 0$ unless $n = m$, an

attribute of a set of density operators that we call "self annihilating". The set of density operators associated with a complete set of spinors is called a "complete set of self annihilating primitive idempotents". Here "primitive" means that the density operators cannot be written as sums of nonzero idempotents. This can be shown by looking at the density matrix in the basis given by $|n\rangle\langle n|$.

Complete sets of self annihilating primitive idempotents are related to the theory of primitive ideals of a Clifford algebra, which is widely discussed in the mathematical literature. An "ideal" is a subalgebra that is closed under multiplication, both on the left and on the right, by all elements of the whole algebra. This sort of theory is more general than just Clifford algebras and may be more familiar to students of mathematics than the approach we will use.

4.1. Primitive Idempotents of Complex Clifford Algebras

Instead of ideals, we will analyze the primitive idempotents of Clifford algebras by seeing how they are generated by complete sets of independent commuting roots of unity. In this context, a root of unity is an operator that squares to 1. If we have a set of roots of unity, and they commute with each other, this is called a "set of commuting roots of unity". "Independent" means that none of the elements of the set can be written as a product of some of the others. If this set is complete in that no more elements of the algebra can be added to it, then we call it a "complete set of independent commuting roots of unity."

From a physics point of view, the commuting roots of unity are operators that are naturally used to classify the complete set of primitive idempotents. In the Pauli algebra, $\hat{z}$ is a root of unity because $\hat{z}^2 = 1$. A complete set of primitive idempotents for the Pauli algebra is $\{(1-\hat{z})/2, (1+\hat{z})/2\}$. These are eigenstates of the operator $\hat{z}$ with eigenvalues ∓ 1. More complicated Clifford algebras are more complicated in terms of primitive idempotents, but not by much.

Let us work with the complex Clifford algebras as they are simpler than the real Clifford algebras. We will need various numbers of canonical basis vectors for our examples. Following our notation, we will begin with $\hat{x}, \hat{y}, \hat{z}$ as our first three canonical basis vectors, and for the next two more complicated Clifford algebras we will add $\hat{t}$ and then $\hat{s}$.[4] This will be enough for our purposes.

To produce a complete set of primitive idempotents, we must first write down as many elements as possible that square to 1 and commute. We label them with the Greek letter ι_n, with n running from 1 to N. It turns out that we can find such a set by starting with any "root of unity", and then keep adding more roots of unity, provided that each commutes with the ones we already have and are independent. Eventually we will run out of Clifford algebra and the set will be complete. Those interested in the proof of this fact are referred to the mathematics literature.

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[4] The fifth canonical basis vector might be thought of as corresponding to proper time, (for example, see [7]) but this is not necessary for the discussion.

In the quantum theory, a phrase that describes a complete set of independent commuting roots of unity is "a complete set of quantum observables". An example of a complete set of quantum observables for a scalar particle in wave mechanics is momentum in the three directions p_x, p_y, and p_z. This paper is devoted to the spin degrees of freedom and we will use the phrase "complete set of independent commuting roots of unity" instead. Of course, as in wave mechanics, there are many different valid choices for a complete set of independent commuting roots of unity. The property "independent" means that one cannot deduce any of the observables from knowledge of the others.

Since the order doesn't matter, we can use our list of canonical basis vectors to produce a sequence of independent commuting roots of unity that will cover all the various cases. One simply stops the sequence at the point where the Clifford algebra runs out of canonical basis vectors. So the structure of the primitive idempotents can be described by a sequence of sets of independent commuting roots of unity. Most of the structure of the set is contained in the size of that set. We will label the set of independent commuting roots of unity that has N elements as U_N.

We begin with the case that the set of independent commuting roots of unity has only a single element. This is the case for the Pauli algebra. The Pauli algebra, as treated in the earlier sections of this paper, is a real Clifford algebra,[5] but in our example here we are dealing with complex Clifford algebras and will treat the Pauli algebra as a complex Clifford algebras.

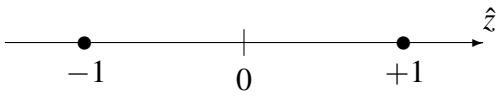

FIGURE 4. Quantum numbers for a Clifford algebra that, like the Pauli algebra, has only one commuting root of unity, $\hat{z}$. The two states are marked with dots.

If the set of independent commuting roots of unity is to have only a single element, we can take this element to be $\hat{z}$. The set of commuting roots is:

$$U_1 = \{\hat{z}\}. \tag{77}$$

This root generates a complete set of self annihilating primitive idempotents that has two elements:

$$\begin{aligned} \rho_{-z} &= (1-\hat{z})/2, \\ \rho_{+z} &= (1+\hat{z})/2. \end{aligned} \tag{78}$$

[5] Note that $\hat{i} = \hat{x}\hat{y}\hat{z}$ is a product of three real elements of the algebra and consequently is in the real Clifford algebra.

The natural operator for distinguishing between these two quantum states is $\hat{z}$. The first state, $(1-\hat{z})/2$ takes an eigenvalue of -1, the second takes an eigenvalue of $+1$. When we plot the two quantum states against the quantum numbers, we have two states symmetrically arranged around the origin. See Fig. (4).

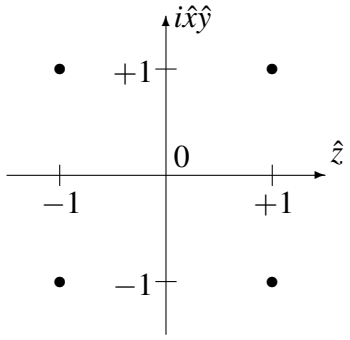

FIGURE 5. Quantum numbers for a Clifford algebra that, like the Dirac algebra, has two commuting roots of unity, $\hat{z}$ and $i\hat{x}\hat{y}$. The four states are marked with dots.

We now consider a case with 2 independent commuting roots of unity. We begin with $U_1 = \{\hat{z}\}$. Since $\hat{x}$ and $\hat{y}$ do not commute with $\hat{z}$, we cannot simply add either to U_1. The product, $\hat{x}\hat{y}$ commutes and is independent of $\hat{z}$, but it squares to -1. So we multiply it by i and the next independent commuting root will be $i\hat{x}\hat{y}$. Our U_2 set of independent commuting roots of unity is:

$$U_2 = \{\hat{z},\ i\hat{x}\hat{y}\}. \tag{79}$$

Note that the above cannot be done in the Pauli algebra because the Pauli algebra assumes that $i = \hat{x}\hat{y}\hat{z}$, and so $i\hat{x}\hat{y} = \hat{z}$ and the above two elements in U_2 would be identical. However, in the Dirac algebra, $\hat{x}\hat{y}\hat{z}$ does not commute with $\hat{t}$ and therefore is not equivalent to i. Thus the above U_2 is a valid set for the Dirac algebra, or larger Clifford algebras.

The two roots of unity in U_2 generate a complete set of primitive idempotents that has four elements:

$$\begin{aligned}
\rho_{-z-ixy} &= (1-\hat{z})(1-i\hat{x}\hat{y})/4, \\
\rho_{-z+ixy} &= (1-\hat{z})(1+i\hat{x}\hat{y})/4, \\
\rho_{+z-ixy} &= (1+\hat{z})(1-i\hat{x}\hat{y})/4, \\
\rho_{+z+ixy} &= (1+\hat{z})(1+i\hat{x}\hat{y})/4.
\end{aligned} \tag{80}$$

We can distinguish between these states with two operators. The obvious selection is $\{\hat{z},\ i\hat{x}\hat{y}\}$. These two operators commute, so their product $i\hat{x}\hat{y}\hat{z}$ is also a root of unity and it commutes with the $\hat{z}$ and $i\hat{x}\hat{y}$. The set $\{\hat{z}, i\hat{x}\hat{y}, i\hat{x}\hat{y}\hat{z}\}$ consists of three commuting roots of unity, but they are not independent. In fact, any two multiply to give the third. Any two of them are independent, and therefore could be used as a complete set of independent commuting roots of unity.

In terms of how the quantum numbers of the states look, the choice in our set of U_2 is not important. There will be two roots of unity, so each state will have two quantum numbers. The quantum numbers will be ± 1, so when we plot them, we will have four states forming a square centered on the origin. See Fig. (5).

The Dirac algebra allows two independent commuting roots of unity which define a set of four primitive idempotents. It is not a coincidence that the Dirac equation supports four particles. The difference between the Dirac equation and the Dirac algebra is that the canonical basis vectors of the Dirac algebra appear with their associated partial derivatives in the Dirac equation. This is called the Dirac operator. In our notation this is:

$$\nabla = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z + \hat{t}\partial_t. \tag{81}$$

The Dirac equation is then $(\nabla + im)\psi = 0$, where m is the mass of the particle.

It is the Dirac operator that is fundamental to physics, not the Dirac algebra, and we have split the Dirac algebra away from the the Dirac operator at our peril. In replacing spinors with operators, it is natural also to replace the spinors of the Dirac equation with operators. This is easily understood in the spinor matrix language and is discussed in [8]. The spinor ψ is replaced by a 4×4 matrix Ψ. The Dirac equation then becomes a matrix equation instead of a spinor equation.

Given a matrix solution to the matrix Dirac equation, we can split it into four spinor solutions by simply taking the four columns of the matrix in turn. We can think of the four spinor solutions as having been obtained by right multiplication by the four kets:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{82}$$

The above four kets are orthonormal. Any set of four orthonormal 4×1 vectors can be thought of as kets, and they generate a complete (Dirac) set of primitive idempotents. And any such set of orthonormal vectors can be used to split a matrix Dirac equation into four spinor solutions. Thus the complete sets of primitive idempotents are related to how one can split a matrix Dirac equation solution into four of the usual Dirac equation solutions.

The fact that the Dirac equation can be upgraded to a matrix equation suggests that we should think of Dirac particles in groups of four. For example, the first column could correspond to the electron (and positron), and the next three columns could be the red, green, and blue up particles. This has been explored in the literature, particularly by Trayling [9] and by Trayling and Baylis [10]. Multiplication on the right by a primitive idempotent corresponds to a Stern-Gerlach experiment that selects a particular fermion type.

Multiplication on the left is more complicated because Dirac algebra elements do not necessarily commute with the Dirac operator, which is on the left of the matrix. If we were able to multiply by a primitive idempotent on the left, we could split a Dirac spinor into its component parts. For specific solutions to the Dirac equation, this can happen. For example, if a solution ψ has no dependence on x or y so that $\partial_x\psi = \partial_y\psi = 0$. Then the root of unity $i\hat{x}\hat{y}$ will commute with ∇ for that solution, and the idempotents $(1 \pm i\hat{x}\hat{y})/2$ will split ψ into two solutions of the Dirac equation. The two solutions will have eigenvalues of ± 1 with respect to the operator $i\hat{x}\hat{y}$. The convention for the Dirac equation is that $i\hat{x}\hat{y}$ is the operator for spin in the z direction, so the two solutions will be

called the spin-up and spin-down solutions.

The natural Dirac operator to represent charge, and therefore define the positron and electron solutions, is $-\hat{t}$. Unfortunately this anticommutes with $\hat{x}$, $\hat{y}$, and $\hat{z}$. To arrange for $\hat{t}$ to commute with ∇, we would require that the wave function $\psi(x,y,z)$ have no dependency on x, y, and z, in other words, a qubit, and this method of defining the electron and positron works for qubits. For general wave functions, one instead splits according to the energy of the solutions. The positive energy solutions are electrons and the negative energy solutions are positrons.

An example of the relation between the above two definitions of the electron and positron may be useful. A common representation of the Dirac matrices has the following choice of γ_0:

$$\hat{t} = \gamma_0 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{83}$$

For the above representation, the plane-wave positive energy spinors one will find in many quantum electrodynamics textbooks are:

$$\psi(p,s) = e^{ip\cdot x}\sqrt{E+m}\begin{bmatrix} \phi^s \\ \frac{\sigma\cdot\vec{p}}{E+m}\phi^s \end{bmatrix}. \tag{84}$$

where ϕ^s is a 2×1 Pauli spinor. If we choose the four-vector p so that it has no spatial dependence, then $\vec{p}=0$, and the above will be of the form:

$$\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \tag{85}$$

where a and b are complex numbers. Under the interpretation of $Q=-\hat{t}$ as a charge operator, the electron (with negative charge), consists of kets of exactly the above form. So if we wish to describe a quantum state with only a spinor (rather than a wave function), this is a situation with no spatial dependence and the correct operator for charge is $-\hat{t}$. This paper is mostly about qubits, which have no spatial dependency, so this definition is appropriate.

To get three commuting roots of unity, we make another root of unity, $\hat{s}\hat{t}$, from the next two canonical basis vectors. A set of three commuting roots is:

$$U_3 = \{\hat{z},\ i\hat{x}\hat{y},\ \hat{s}\hat{t}\}. \tag{86}$$

The above requires s, one more dimension than the four dimensions of the Dirac equation. The three roots generate a complete set of 8 primitive idempotents:

$$\rho_{\pm z\pm ixy\pm st} = (1\pm\hat{z})(1\pm i\hat{x}\hat{y})(1\pm\hat{s}\hat{t})/8, \tag{87}$$

where the $\pm$ are independent. These states require three operators to distinguish; they

form the vertices of a cube. The generalization to yet more complicated Clifford algebras is clear. The quantum numbers form a hypercube. Adding $2M-1$ or $2M$ hidden dimensions to the usual 4 will result in $2+M$ commuting roots of unity, and a complete set of primitive idempotents with size 2^{2+M}.

4.2. The Elementary Fermions

We now approach the question of how a Clifford algebra can be used to model the elementary fermions. Our task is to plot the elementary particles according to their quantum numbers, and look for a hypercubic structure. Since the four states in the Dirac algebra include antiparticles, we will have to plot the antiparticles as well. In addition, since the standard model deals with the chiral states, we will plot the quantum numbers of the handed states rather than the spin up and down states.

TABLE 1. Weak hypercharge (t_0) and weak isospin (t_3) quantum numbers for the first generation particles and antiparticles.

	t_0	t_3		t_0	t_3
ν_L	$-1/2$	$+1/2$	$\bar{\nu}_R$	$+1/2$	$-1/2$
ν_R	0	0	$\bar{\nu}_L$	0	0
d_L	$+1/6$	$-1/2$	$\bar{d}_R$	$-1/6$	$+1/2$
d_R	$-1/3$	0	$\bar{d}_L$	$+1/3$	0
e_L	$-1/2$	$-1/2$	$\bar{e}_R$	$+1/2$	$+1/2$
e_R	-1	0	$\bar{e}_L$	$+1$	0
u_L	$+1/6$	$+1/2$	$\bar{u}_R$	$-1/6$	$-1/2$
u_R	$+2/3$	0	$\bar{u}_L$	$-2/3$	0

In the qubit model, the four complex degrees of freedom of the Dirac electron are split into four separate particles. We will call them the left and right handed electron and positron. Counting this way, there are a total of 96 elementary fermions:

$$\begin{array}{ccccccc} Q/\bar{Q} & \text{Left/Right} & \text{Leptons} & \text{Quarks} & \text{gen} & & \\ \hline 2 & 2 & (2+ & 2\times 3) & \times 3 & = & 96. \end{array} \tag{88}$$

In the above the two left most numbers are the 4 degrees of freedom in a Dirac spinor. The two leptons are the electron and neutrino. There are two quarks, and each comes in three colors. Finally, there are three generations, giving a total of 96 elementary fermions.

This large number is suggestive that the states are bound states of some deeper particle structure. It would be convenient if there were a power of 2 elementary fermions as this would fit perfectly into the primitive idempotent structure of a Clifford algebra. The difference here is the three particle generations. For the moment we will ignore them by supposing that they are some sort of excitation, and look only at the first generation. This reduces us to 32 states. These would be further reduced to 16 if we temporarily ignored the different colors of quarks.

The weak hypercharge and weak isospin numbers for the first generation are shown in Table (1). The resulting drawing, Fig. (6), shows that the leptons form the vertices of

a cube. Other quantum numbers, such as electric charge, neutral charge or hypercharge, depend linearly on these quantum numbers and consequently will produce similar drawings with various distortions. The cubic structure is evident.

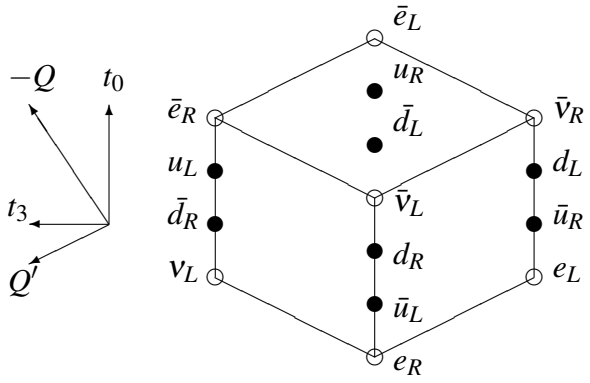

FIGURE 6. Weak hypercharge, t_0, and weak isospin, t_3, quantum numbers plotted for the first generation standard model quantum states. Leptons are hollow circles and quarks ($\times 3$) are filled circles. Electric charge, Q, and neutral charge, Q' also shown.

In Fig. (6), the leptons are in the corners of a cube, and so they match the states expected from a complex Clifford algebra with five or six canonical basis vectors. The quarks show up in columns between pairs of leptons. Each quark represents a triplet, so the state count for one of these columns is $(1,3,3,1)$. This pattern of states is suspiciously reminiscent of a line in Pascal's triangle, and suggests that the fermions are composed of three preons, ρ, each as:

$$\begin{array}{rcccccccc} & & \bar{e}_R & & 3u_L & & 3\bar{d}_R & & \nu_L, \\ (\rho_{\bar{e}R}+\rho_{\nu L})^3 & = & (\rho_{\bar{e}R})^3 & + & 3(\rho_{\bar{e}R})^2(\rho_{\nu L}) & + & 3(\rho_{\bar{e}R})(\rho_{\nu L})^2 & + & (\rho_{\nu L})^3, \end{array} \tag{89}$$

using the left most column as an example. In the above, the preon states contributing to the fermion are shown before the fermion. The $\bar{e}_R$ is composed of three $\rho_{\bar{e}R}$ states.

By the Pauli exclusion principle, these states must be distinguishable. Following the example of the quarks, we suppose that the $\rho_{\bar{e}R}$ must come in three "precolors", and the fermions are all precolor singlets. We will designate the precolors by 1, 2 and 3 so that the $\rho_{\nu R}$ preon comes in three precolors, $\rho_{\nu R1}$, $\rho_{\nu R2}$, and $\rho_{\nu R3}$. See Fig. (7) for the composition of all eight chiral fermions in the left most column.

In this preon model, the quarks are also composed of three mixed preons. As with the leptons, the three preons must have different precolor, but one of the preons is distinguishable from the other two by more than just its precolor. Thus the quark states are precolor neutral. Color appears as that part of the precolor that depends on the difference in weak hypercharge of the component preons.

The binding together of three preons in this manner reminds one of how quarks bind together. An analogy to the eight states shown in Fig. (7) would be eight states obtained by binding together three quarks taken from two different types. To discuss this, let's consider how one can similarly combine three up or down quarks to make eight baryons.

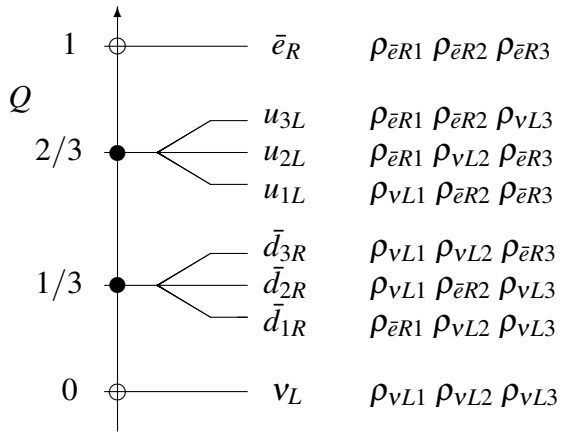

FIGURE 7. A column of quarks and leptons shown as bound states of three preons each. Q is electric charge. The three colors and precolors are marked 1, 2, and 3.

The lowest mass state that can be made from three up quarks is the $\Delta^{++}(1232)P_{33}$. Switching the quarks from up to down one at a time gives three more $I = 3/2$ states:

$$\begin{array}{cccc} & ddu & duu & \\ ddd & dud & udu & uuu \\ & udd & uud & \\ \hline \Delta^- & \Delta^0 & \Delta^+ & \Delta^{++} \end{array} \tag{90}$$

The ddu, dud, and udd differ only in the color of the u quark. We cannot distinguish between these, so the Δ^0 state is treated as a symmetric combination of these three states, $(ddu + dud + udd)/\sqrt{3}$. Similarly for the Δ^+.

5. FINITE GROUPS AND QUANTUM NUMBERS

The pure density matrices can be characterized as all those things that can be created from normalized spinors. For example, in the Pauli algebra, a normalizes spinor takes the form:

$$\begin{pmatrix} a \\ b \end{pmatrix}, \tag{91}$$

where $|a|^2 + |b|^2 = 1$. The associated density matrix is:

$$\begin{pmatrix} |a|^2 & a^* b \\ b^* a & |b|^2 \end{pmatrix}, \tag{92}$$

again subject to $|a|^2 + |b|^2 = 1$. But this definition is written in terms of spinors and uses a particular choice of representation. We now look at how this definition could be written more generally.

The pure density matrices are idempotent, that is, they satisfy $\rho^2 = \rho$. If we were to define the valid quantum states according to the idempotency relation, it turns out that we would end up with too many objects. First, $\rho = 0$, and $\rho = 1$ satisfy $\rho^2 = \rho$, so we must eliminate these. We can eliminate these by requiring that the density matrix be primitive. However, there will still be unwanted matrices included, for example:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0.5(1 + \hat{x} + i\hat{y} + \hat{z}), \tag{93}$$

is a primitive idempotent. To eliminate these, we can require that the states be Hermitian. In geometric language, the above is not Hermitian because of the i multiplying $\hat{y}$ in its geometric form.

This definition of the pure density matrix states of the Pauli algebra generalizes to any Clifford algebra; the quantum density operator states, (the operators that represent quantum measurements in the operator algebra) are just the Hermitian primitive idempotents of the algebra. We can interpret the zero idempotent as the state with no particles, and we can interpret the various non primitive idempotents as states where two or more particles are at the same point, for example, a spin up electron and a spin down electron. Thus all the idempotents of the algebra get interpretations in terms of particles.

5.1. Finite Groups and Idempotency Equations

Suppose we have two Hermitian density operators in the Pauli algebra, say:

$$\begin{aligned} \rho_1 &= A_1 + B_1\hat{x} + C_1\hat{y} + D_1\hat{z}, \\ \rho_2 &= A_2 + B_2\hat{x} + C_2\hat{y} + D_2\hat{z}. \end{aligned} \tag{94}$$

where A_n, B_n, C_n, and D_n are real numbers (which says that they are Hermitian. The complex Pauli algebra is spanned by the basis elements $\{1, \hat{x}, \hat{y}, \hat{z}\}$, so Pauli algebra element can be written in this basis. In particular, the product of ρ_1 and ρ_2 can be so written. This breaks the product up into four parts:

$$\begin{aligned} (\rho_1\rho_2)_1 &= A_1A_2 + B_1B_2 + C_1C_2 + D_1D_2, \\ (\rho_1\rho_2)_x &= A_1B_2 + B_1A_2 + iC_1D_2 - iD_1C_2, \\ (\rho_1\rho_2)_y &= A_1C_2 - iB_1D_2 + C_1A_2 + iD_1B_2, \\ (\rho_1\rho_2)_z &= A_1D_2 + iB_1C_2 - iC_1B_2 + D_1A_2. \end{aligned} \tag{95}$$

There are 4 elements in ρ_1 and 4 more in ρ_2. Each of the 16 possible pairings of these elements appears exactly once on the right hand side of the above. They are multiplied by 1, i, or $-i$. And the four equations list the four components of the product on the left hand side. Furthermore, since the pure density operator states of the Pauli algebra are the Hermitian primitive idempotents, these four equations contain everything there is to know about the quantum states of the Pauli algebra.

When one sees a set of equations like that of Eq. (95), one is reminded of a finite group. Were it not for the $\pm i$, one could think of this as a set of equations defining a

group of four elements as follows:

$$\begin{array}{c|cccc} & 1 & x & y & z \\ \hline 1 & 1 & x & y & z \\ x & x & 1 & z & y \\ y & y & z & 1 & x \\ z & z & y & x & 1 \end{array} . \tag{96}$$

This the finite group $Z_2 \times Z_2$. The group we really want looks like this, but needs to have complex phases given for some of the products:

$$\begin{array}{c|cccc} & 1 & x & y & z \\ \hline 1 & 1 & x & y & z \\ x & x & 1 & +iz & -iy \\ y & y & -iz & 1 & +ix \\ z & z & +iy & -ix & 1 \end{array} . \tag{97}$$

The above does define a finite group, but it now has 16 elements, namely $\{1,x,y,z\} \times \{+1,-1,+i,-i\}$. The multiplication rules for these can be derived from the above by multiplying the columns by the appropriate phase. For example, $(+i\hat{x})(-\hat{y}) = (-i)(i\hat{z}) = i\hat{z}$.

Here we have taken the idempotency relation, written it in geometric form for Hermitian elements, and obtained a finite group from the result. We now consider the reverse operation, that is, given a finite group, we obtain an idempotency relation from it. Our motivation for doing this is to see if there is a finite group that will give the quantum numbers of the observed elementary fermions as solutions to a primitive idempotency problem.

If we begin with the abbreviated group definition given in Eq. (97), it is clear that we will get the correct idempotency equations. If we expand that group definition to the larger group with 16 elements, that is, we define the group multiplication for the complex phases, we will instead get 16 equations in 16 unknowns. These 16 equations can then be reduced to the usual 4 equations.

5.2. The Permutation Group on 3 Elements

The basic idea of perturbation theory is that the force between elementary particles can be broken up into several parts. One finds an exact solution for part of the force, then adds the more difficult to solve parts as a perturbation. The strength of the perturbation is given by the coupling constant associated with the force. The result is a series of more and more complicated diagrams as the additional force is included in more complicated interactions.

The series of diagrams are converted into a Taylor series in the coupling constant. The method fails when the strength of the coupling constant exceeds the radius of convergence of the Taylor series. Of the coupling constants of the elementary particles,

only the color force is sufficiently large (about 1/3) that it presents an impossible barrier to the usual methods of perturbation theory.

When the coupling constant is sufficiently small, one can ignore the more complicated diagrams as they contribute very little to the overall amplitude. We will be doing the opposite of this. Our calculations will take into account the very complicated contributions to an amplitude but will ignore the simple ones.

In the qubit representation of a quantum state there is no position, time, momentum or energy information. We will work in a similar situation. To justify this, the reader can suppose that the wave state of the chiral fermion being analyzed is approximately stable and has become so spread out that it is approximately a constant function of position. While we do not worry about the details of position and time, we will be concerned with the order in which things happen.

And we are concerned with bound states. We will suppose that these states are bound for a very long time, relative to the simple interactions, so the Feynman diagrams that describe that very long duration bound state will be very complicated Feynman diagrams. The simple diagrams appropriate to scattering theory have no interest to us here.

We will discuss a generic chiral fermion f that can stand for any of the chiral fermions. We suppose that it is a bound state of three colored "valence" preons which we will label f_R, f_G, and f_B. This is an unimportant change from our previous notation of f_1, f_2, f_3, We are making the change so that the symbols we use to represent permutations will be more natural. As with the standard model theory of how three quarks make up a baryon, we suppose that the valence preons are always present, though their form is constantly changing, and that they are typically accompanied by "sea" preons and force bosons.

We imagine ourselves watching this collection of valence and sea particles interacting. In quantum mechanics, anything that can happen, no matter how unlikely, will eventually happen. Accordingly, if we wait long enough, we will find a moment, t_0, when the sea has completely vanished and the only particles present are the three valence preons. These are particularly simple moments in the evolution of the bound state.

The Feynman diagrams that connect between these states are convenient to analyze because they all have the same three inputs and three outputs. We can define a multiplication and addition on these Feynman diagrams. Addition is the usual action of describing alternative paths. To multiply two diagrams, we take the output states of one and hook them up to the input states of the other.

To hook the outputs of one diagram up to the inputs of the other, one would normally require that the red output be hooked up to the red input, green to green, and blue to blue. On the other hand, the gluons carry color charge so the various diagrams will permute the colors. However, the colors are arbitrary, so we can always permute the input colors on a diagram to match the output colors of the previous diagram. Therefore we will not worry about matching the colors up correctly.

Each of these diagrams is characterized by the propagators of the three valence particles. These propagators connect the input states to the output states. The interactions that we will be concerned with will be gluon emission and absorption. The gluons can create sea quarks, but these can interact with the valence quarks only through more gluons. Consequently, none of this complication appears in the inputs and outputs of our Feynman diagrams, so we do not have to worry about matching these up.

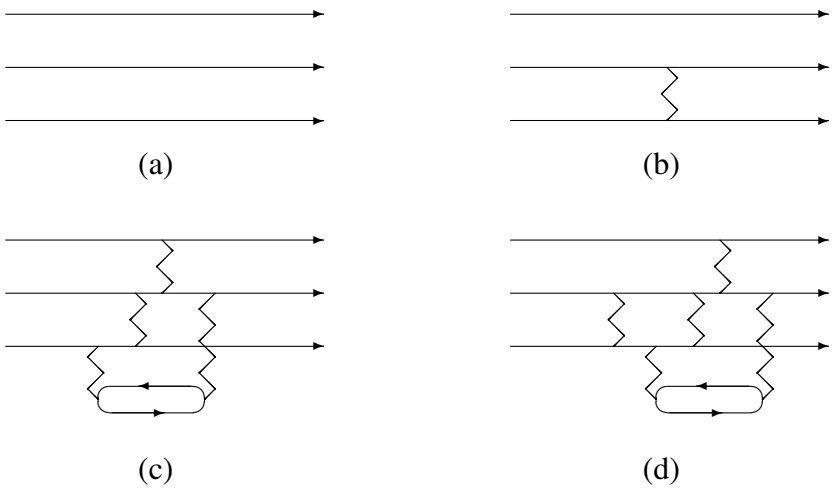

FIGURE 8. Examples of Feynman diagrams. (a) the unit; (b) and (c) primes; (d) a composite diagram, the product of (b) and (c).

The identity is the trivial diagram, the one where the inputs directly connect to the outputs. With these definitions, these very special Feynman diagrams form a field, and we can define units, primes, and composites.

5.3. Consistency Equations

In the language of a mathematical field, the trivial Feynman diagram is a "unit". The other diagrams are either composite or prime, depending on whether or not they can be written as a product of non trivial Feynman diagrams. See Fig. (8) for an illustration showing the unit diagram, two prime diagrams, and their composite product.

Suppose we have a very complicated Feynman diagram as would be suitable for a long lived bound state, say for a duration τ. We would like to write this diagram as the product of two diagrams corresponding to durations $\tau/2$. For sufficiently long times, it seems reasonable to assume that almost all Feynman diagrams will be able to be approximately divided in this way. We will make this assumption.

There are six possible permutations of three objects, so we will classify our diagrams into six classes. In summing Feynman diagrams, we will compute sums for each of these possible permutations, and label the sums as follows:

$$\begin{array}{rcll} F_I &=& (R,G,B) \to (R,G,B) & \text{identity,} \\ F_J &=& (R,G,B) \to (G,B,R) & \text{rotate 1,} \\ F_K &=& (R,G,B) \to (B,R,G) & \text{rotate 2,} \\ F_R &=& (R,G,B) \to (R,B,G) & \text{swap GB,} \\ F_G &=& (R,G,B) \to (B,G,R) & \text{swap RB,} \\ F_B &=& (R,G,B) \to (G,R,B) & \text{swap RG,} \end{array} \tag{98}$$

Each of these is a complex number.

The permutations of three objects forms a group under composition:

	I	J	K	R	G	B
I	I	J	K	R	G	B
J	J	K	I	B	R	G
K	K	I	J	G	B	R
R	R	G	B	I	J	K
G	G	B	R	K	I	J
B	B	R	G	J	K	I

. (99)

The above rules show how a given permutation can be obtained six different ways as a product of two permutations. For example, R appears as IR, JG, KB, RI GK, and BJ.

In general, when we sum the Feynman diagrams, the sums will depend on the duration τ, of the diagrams. We make the assumption that for sufficiently large τ, the sums converge, and therefore we will have the same sums for the τ Feynman diagrams as the $\tau/2$ diagrams.

We are assuming that the Feynman diagrams of duration τ are composite and can be split into Feynman diagrams of duration $\tau/2$. Furthermore, the sums for these two durations are assumed to be the same giving us:

$$\begin{aligned}
F_I &= F_IF_I + F_JF_K + F_KF_J + F_RF_R + F_GF_G + F_BF_B, \\
F_J &= F_IF_J + F_JF_I + F_KF_K + F_RF_G + F_GF_B + F_BF_R, \\
F_K &= F_IF_K + F_JF_J + F_KF_I + F_RF_B + F_GF_R + F_BF_G, \\
F_R &= F_IF_R + F_JF_G + F_KF_B + F_RF_I + F_GF_K + F_BF_J, \\
F_G &= F_IF_G + F_JF_B + F_KF_R + F_RF_J + F_GF_I + F_BF_K, \\
F_B &= F_IF_B + F_JF_R + F_KF_G + F_RF_K + F_GF_J + F_BF_I,
\end{aligned} \tag{100}$$

The left hand F_χ are the sums for the Feynman diagrams of duration τ, while the F_χ on the right hand side are the sums for the diagrams of duration $\tau/2$. Under our assumption, these are equal; this gives us six quadratic equations in six unknowns.

In the above six equations, each F_χ appears once on the left, and twelve times on the right hand side, multiplying each of the other terms on the left or right. Therefore, if we add all six equations together, we find that

$$\sum F_\chi = \left(\sum F_\chi\right)^2. \tag{101}$$

There are two solutions to this equation, and we can therefore classify the solutions to the six equations as follows:

$$\begin{aligned}
F_I + F_J + F_K + F_R + F_G + F_B &= 0, \text{ or} \\
&= 1.
\end{aligned} \tag{102}$$

The above amounts to the trivial representation of the permutation group on three elements, the representation that maps all the elements to unity. It will turn out that the mass interaction conserves this quantum number.

Define G_0, G_1, and G_2 as follows:

$$\begin{array}{rcl} G_I &=& F_I + F_R + F_G + F_B, \\ G_J &=& F_J + F_R + F_G + F_B, \\ G_K &=& F_K + F_R + F_G + F_B. \end{array} \tag{103}$$

Then G_χ satisfy the following equations:

$$\begin{array}{rcl} G_I &=& G_I G_I + 2 G_J G_K, \\ G_J &=& G_K G_K + 2 G_I G_J, \\ G_K &=& G_J G_J + 2 G_I G_K. \end{array} \tag{104}$$

This can be written as an idempotency equation for a 3×3 circulant matrix:

$$\begin{pmatrix} G_I & G_J & G_K \\ G_K & G_I & G_J \\ G_J & G_K & G_I \end{pmatrix}^2 = \begin{pmatrix} G_I & G_J & G_K \\ G_K & G_I & G_J \\ G_J & G_K & G_I \end{pmatrix}. \tag{105}$$

In the above, the G_χ form the usual representation of the even permutation group of 3 elements in 3×3 matrices. For example, the action of G_1 on the vector (A, B, C) is to perform the permutation:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} B \\ C \\ A \end{pmatrix}. \tag{106}$$

There are three G_χ quadratic equations Eq. (104), and there have three unknowns. The solutions are:

$$\begin{array}{ccc} 3G_I & 3G_J & 3G_K \\ \hline 0 & 0 & 0 \\ 1 & +1 & +1 \\ 1 & +\zeta^{+8} & +\zeta^{-8} \\ 1 & +\zeta^{-8} & +\zeta^{+8} \\ 2 & -1 & -1 \\ 2 & -\zeta^{+8} & -\zeta^{-8} \\ 2 & -\zeta^{-8} & -\zeta^{+8} \\ 3 & 0 & 0 \end{array}, \tag{107}$$

where $\zeta = \exp(i\pi/12)$. These are easier to solve than the six quadratic equations of Eq. (100).

The six quadratic equations in six unknowns can be rearranged into the more convenient form:

$$\begin{array}{rcl} F_I &=& F_I^2+2F_JF_K+(F_R^2+F_G^2+F_B^2), \\ (F_R+F_G+F_B)^2 &=& (F_I+F_J+F_K)(1-(F_I-F_J-F_K)), \\ 0 &=& (F_J-F_K)(1+F_J+F_K-2F_I), \\ (1-3F_I+(F_I+F_J+F_K))F_R &=& (F_R+F_G+F_B)(F_J+F_K), \\ (1-3F_I+(F_I+F_J+F_K))F_G &=& (F_R+F_G+F_B)(F_J+F_K), \\ (1-3F_I+(F_I+F_J+F_K))F_B &=& (F_R+F_G+F_B)(F_J+F_K). \end{array} \tag{108}$$

If one chooses $F_I = 1/2$ and $F_J = -F_K$, one finds that the last four of the above equations are satisfied for any values of F_R, F_G, and F_B. One finds that there are an infinite number of solutions with two parameters. The remaining equations become:

$$\begin{array}{rcl} 1/2 &=& \pm(F_R+F_G+F_B), \\ F_J^2 &=& -1/8+(F_R^2+F_G^2+F_B^2)/2. \end{array} \tag{109}$$

The two choices of sign in the first equation are symmetric in the second equation. That is, one can take a solution (F_R,F_G,F_B) to another solution $(-F_R,-F_G,-F_B)$. Similarly, F_J can be taken to $-F_J$.

We will parameterize these solutions around $F_R = F_G = F_B = \pm 1/6$. There are four solutions, depending on the choice of $\pm$ in Eq. (109) and the choice of sign for F_J and F_K:

$$\begin{array}{cccccc} F_I & F_J & F_K & F_R & F_G & F_B \\ \hline 1/2 & +\gamma & -\gamma & +1/6+\alpha & +1/6+\beta & +1/6-\alpha-\beta \\ 1/2 & -\gamma & +\gamma & +1/6+\alpha & +1/6+\beta & +1/6-\alpha-\beta \\ 1/2 & +\gamma & -\gamma & -1/6+\alpha & -1/6+\beta & -1/6-\alpha-\beta \\ 1/2 & -\gamma & +\gamma & -1/6+\alpha & -1/6+\beta & -1/6-\alpha-\beta \end{array} , \tag{110}$$

where $\gamma = \sqrt{\alpha^2+\beta^2+\alpha\beta-1/12}$. This complication of an infinite number of solutions only appears in the solutions with $F_I = 1/2$. The other cases are finite, and all have $F_R = F_G = F_B$.

By choosing various values of α and β, we obtain various interesting solutions to the six quadratic equations. Some examples of these solutions are listed here:

$$\begin{array}{cccccc} F_I & F_J & F_K & F_R & F_G & F_B \\ \hline 1/2 & 0 & 0 & 0 & 0 & \pm 1/2 \\ 1/2 & 0 & 0 & \pm 1/3 & \pm 1/3 & \mp 1/6 \\ 1/2 & +i\sqrt{3}/2 & -i\sqrt{3}/2 & \pm 1/6 & \pm 1/6 & \pm 1/6 \\ 1/2 & -i\sqrt{3}/2 & +i\sqrt{3}/2 & \pm 1/6 & \pm 1/6 & \pm 1/6 \end{array} . \tag{111}$$

where the $\pm$ indicate whether $F_R+F_G+F_B = \pm 1/2$. The first two cases, which pick out F_B, can be rotated around to pick out F_R or F_G.

When $F_0 \neq 1/2$, there are eight more solutions to the six quadratic equations having one solution for F_0 in $\{0,1\}$, two solutions when $F_0 = 1/6$, and three solutions for F_0 in $\{1/3,2/3\}$. These solutions are discrete. Putting the various solutions in a convenient

order, we have:

	$\sum F_\chi$	F_I	F_J	F_K	F_R	F_G	F_B
$S_{1/2-\pm}$	0	$1/2$	$\pm\gamma$	$\mp\gamma$	$-1/6+\alpha$	$-1/6+\beta$	$-1/6-\alpha-\beta$
S_0	0	0	0	0	0	0	0
$S_{1/6-}$	0	$1/6$	$1/6$	$1/6$	$-1/6$	$-1/6$	$-1/6$
$S_{1/3\pm}$	0	$1/3$	$\zeta^{\pm 8}/3$	$\zeta^{\mp 8}/3$	0	0	0
$S_{1/2+\pm}$	1	$1/2$	$\pm\gamma$	$\mp\gamma$	$+1/6+\alpha$	$+1/6+\beta$	$+1/6-\alpha-\beta$
S_1	1	1	0	0	0	0	0
$S_{1/6+}$	1	$1/6$	$1/6$	$1/6$	$+1/6$	$+1/6$	$+1/6$
$S_{2/3\pm}$	1	$2/3$	$\zeta^{\pm 4}/3$	$\zeta^{\mp 4}/3$	0	0	0
$S_{2/3}$	0	$2/3$	$-1/3$	$-1/3$	0	0	0
$S_{1/3}$	1	$1/3$	$1/3$	$1/3$	0	0	0

(112)

again where $\zeta = \exp(i\pi/12)$ as before, $\gamma = \sqrt{\alpha^2+\beta^2+\alpha\beta-1/12}$, and we have labeled the solutions according to their F_0 value, the sign of $F_R+F_G+F_B$, and or the sign of the power of ζ.

5.4. Weak hypercharge and weak isospin

Six of the solutions of Eq. (112) are similar in that they are zero in F_R, F_G, and F_B. The other four share pairs of F_I and come in two doublets with $F_R+F_R+F_B = \pm 1/2$. This form reminds one of the weak isospin quantum numbers of the elementary fermions. In addition, the values of F_I match the weak hypercharge quantum numbers, if one takes into account the fact that antiparticles carry negated quantum numbers.

There are only eight fundamental chiral fermions, but there are ten solutions given above, fourteen if different signs are counted. We interpret the fourteen solutions as being the complete set of possibilities. Only eight are used by the elementary fermions. But the mesons and baryons will use all fourteen as will be discussed later in this paper.

Only the $F_I = 0$, and $F_I = 1$ solutions are unique. These solutions have to be associated with the right handed neutrino and the right handed electron, respectively. The others have to be assigned somewhat arbitrarily. We put $\alpha = \beta = 0$ so that $F_R = F_G = F_B$ and

assign:

	t_0	t_3	S_χ	$\sum F_\chi$	F_I	$3F_J$	$3F_K$	$3F_R$
ν_L	$-1/2$	$-1/2$	$S_{1/2-+}$	0	$1/2$	$+i\sqrt{3}/2$	$-i\sqrt{3}/2$	$-1/2$
ν_R	0	0	S_0	0	0	0	0	0
d_L	$1/6$	$-1/2$	$S_{1/6-}$	0	$1/6$	$1/2$	$1/2$	$-1/2$
d_R	$-1/3$	0	$S_{1/3\pm}$	0	$1/3$	$-1/2\pm i\sqrt{3}/2$	$-1/2\mp i\sqrt{3}/2$	0
e_L	$-1/2$	$+1/2$	$S_{1/2++}$	1	$1/2$	$+i\sqrt{3}/2$	$-i\sqrt{3}/2$	$+1/2$
e_R	-1	0	S_1	1	1	0	0	0
u_L	$1/6$	$+1/2$	$S_{1/6+}$	1	$1/6$	$1/2$	$1/2$	$+1/2$
u_R	$2/3$	0	$S_{2/3\pm}$	1	$2/3$	$+1/2\mp i\sqrt{3}/2$	$+1/2\pm i\sqrt{3}/2$	0

(113)

In the above we have suppressed the differences between particles and antiparticles. We will have to include them later, but for now, the assignments are more obvious with this suppressed.

As noted before, the left and right handed particles share the same $\sum F_\chi$. In addition, the action of the mass interaction in converting from left to right handed and back is of the following form:

	F_I	F_J	F_K	F_R	F_G	F_B
L to R	$+1/2$	$+i\sqrt{3}/6$	$-i\sqrt{3}/6$	$-1/6$	$-1/6$	$-1/6$
R to L	$-1/2$	$-i\sqrt{3}/6$	$+i\sqrt{3}/6$	$+1/6$	$+1/6$	$+1/6$

(114)

These are $\pm S_{1/2-}$. That is, the mass interaction corresponds to a change in F_χ that is also an element of the F_χ.

In assigning the elementary fermions to solutions to the F_χ equations as in Eq. (113), we are associating these solutions with particles. In showing that the mass interaction causes a change in the F_χ that corresponds to a solution to these same equations, we are showing that the Higgs particle, or its equivalent, satisfies the same equation and can also can be thought of as a particle.

We organized Eq. (11) so that the left and right handed parts of the same particle were paired together. We can also organize the table so that particles that are related by the weak force are paired together. We find four pairs of two pairs of particles, the doublets

of the weak force, and four singlets:

	t_0	t_3	S_χ	$\sum F_\chi$	F_I	$3F_J$	$3F_K$	$3F_R$
ν_L	$-1/2$	$-1/2$	$S_{1/2-+}$	0	$1/2$	$+i\sqrt{3}/2$	$-i\sqrt{3}/2$	$-1/2$
e_L	$-1/2$	$+1/2$	$S_{1/2++}$	1	$1/2$	$+i\sqrt{3}/2$	$-i\sqrt{3}/2$	$+1/2$
d_L	$1/6$	$-1/2$	$S_{1/6-}$	0	$1/6$	$1/2$	$1/2$	$-1/2$
u_L	$1/6$	$+1/2$	$S_{1/6+}$	1	$1/6$	$1/2$	$1/2$	$+1/2$
ν_R	0	0	S_0	0	0	0	0	0
e_R	-1	0	S_1	1	1	0	0	0
d_R	$-1/3$	0	$S_{1/3\pm}$	0	$1/3$	$-1/2\pm i\sqrt{3}/2$	$-1/2\mp i\sqrt{3}/2$	0
u_R	$2/3$	0	$S_{2/3\pm}$	1	$2/3$	$+1/2\mp i\sqrt{3}/2$	$+1/2\pm i\sqrt{3}/2$	0

(115)

From the above, we see that we can characterize the weak interaction according to a change in the F_χ quantum numbers as follows:

		F_I	F_J	F_K	F_R	F_G	F_B
-1/2 to +1/2	W^+	0	0	0	$+1/3$	$+1/3$	$+1/3$
+1/2 to -1/2	W^-	0	0	0	$-1/3$	$-1/3$	$-1/3$

(116)

All the particle pairs distinguished by this set of quantum numbers are related by the weak force and none of the others are so related.

The quantum numbers listed in Eq. (116) are not listed in the table of solutions to the six quadratic equations given in Eq. (112). We must therefore suppose that the weak particle is not an elementary colorless particle, but instead is composed of at least two colorless parts. Somewhat arbitrarily, we will assume that there are two parts and label them as left and right. Examining the table of solutions, we find two possible cases:

		F_I	F_J	F_K	F_R	F_G	F_B
W^+		0	0	0	$1/3$	$1/3$	$1/3$
W_L^+	$S_{1/6+}$	$+1/6$	$+1/6$	$+1/6$	$1/6$	$1/6$	$1/6$
W_R^+	$\bar{S}_{1/6-}$	$-1/6$	$-1/6$	$-1/6$	$1/6$	$1/6$	$1/6$
W_L^+	$S_{1/2+\pm}$	$+1/2$	$\pm\gamma$	$\mp\gamma$	$1/6$	$1/6$	$1/6$
W_R^+	$\bar{S}_{1/2-\pm}$	$-1/2$	$\mp\gamma$	$\pm\gamma$	$1/6$	$1/6$	$1/6$

(117)

Gauge bosons that are elastic, that is, that do not carry any charge and so can be absorbed by a particle without changing the quantum numbers of the particle, have must have all their F_χ zero. Such particles include the Z and photon. This is also a solution to the F_χ equations.

Thus the weak, electromagnetic, and gravitational forces can be described as simple solutions to the F_χ equations. Only the weak force requires an assumption that two colorless particles are involved, but only the $W^\pm$ (and Z) have mass. This gives us some

hope that the classification of particles according to their F_χ quantum numbers will be useful in understanding the makeup of the elementary particles.

The F_χ are vectors of six complex numbers and we've been using the natural addition defined on those vectors. Since the F_χ are defined from permutations, these vectors of complex numbers also possess a natural product. Given two vectors,

$$\begin{array}{rcl} A &=& (a_I, a_J, a_K, a_R, a_G, a_B), \\ B &=& (b_I, b_J, b_K, b_R, b_G, b_B), \end{array} \tag{118}$$

where the a_χ and b_χ are complex numbers, the natural product is defined as:

$$AB = (p_I, p_J, p_K, p_R, p_G, p_B), \tag{119}$$

where p_χ are defined by the relations between $\{I,J,K,R,G,B\}$. That is,

$$\begin{array}{rcl} p_I &=& a_I b_I + a_J b_K + a_K b_J + a_R b_R + a_G b_G + a_B b_B, \\ p_J &=& a_I b_J + a_J b_I + a_K b_K + a_R b_G + a_G b_B + a_B b_R, \\ p_K &=& a_I b_K + a_J b_J + a_K b_I + a_R b_B + a_G b_R + a_B b_G, \\ p_R &=& a_I b_R + a_J b_G + a_K b_B + a_R b_I + a_G b_K + a_B b_J, \\ p_G &=& a_I b_G + a_J b_B + a_K b_R + a_R b_J + a_G b_I + a_B b_K, \\ p_B &=& a_I b_B + a_J b_R + a_K b_G + a_R b_K + a_G b_J + a_B b_I. \end{array} \tag{120}$$

With this definition of multiplication, the defining equations for the F_χ, Eq. (100), become an idempotency relation. That is, they can be written as $F = FF$.

With the definition of a multiplication rule, the multiplication table for the F_χ is of interest. It turns out that products of F_χ are F_χ, as one expects for the idempotency table for a matrix group. The product of S_0 with anything is S_0, and the product of S_χ with S_1 is S_χ. Leaving off the multiplicative and additive identities, and suppressing the "S" for

brevity, the table[6] is:

	16^-	16^+	13	13^-	13^+	12^{--}	12^{+-}	12^{-+}	12^{++}	23	23^-	23^+
16^-	16^-	0	16^-	0	0	16^-	0	16^-	0	0	16^-	16^-
16^+	0	16^+	16^+	0	0	0	16^+	0	16^+	0	16^+	16^+
13	16^-	16^+	13	0	0	16^-	16^+	16^-	16^+	0	13	13
13^-	0	0	0	13^-	0	13^-	13^-	0	0	13^-	13^-	0
13^+	0	0	0	0	13^+	0	0	13^+	13^+	13^+	0	13^+
12^{--}	16^-	0	16^-	13^-	0	12^{--}	13^-	16^-	0	13^-	12^{--}	16^-
12^{+-}	0	16^+	16^+	13^-	0	13^-	12^{+-}	0	16^+	13^-	12^{+-}	16^+
12^{-+}	16^-	0	16^-	0	13^+	16^-	0	12^{-+}	13^+	13^+	16^-	12^{-+}
12^{++}	0	16^+	16^+	0	13^+	0	16^+	13^+	12^{++}	13^+	16^+	12^{++}
23	0	0	0	13^-	13^+	13^-	13^-	13^+	13^+	23	13^-	13^+
23^-	16^-	16^+	13	13^-	0	12^{--}	12^{+-}	16^-	16^+	13^-	23^-	13
23^+	16^-	16^+	13	0	13^+	16^-	16^+	12^{-+}	12^{++}	13^+	13	23^+

(121)

From the above table, the primitive structure of the idempotents can be determined.

As with the F_χ, the solutions for the G_χ equations can also be expressed as an idempotency relation. There are only three complex degrees of freedom in the G_χ. These three complex degrees of freedom can be used to define a 3×3 circulant matrix:

$$G = \begin{pmatrix} G_I & G_J & G_K \\ G_K & G_I & G_J \\ G_J & G_K & G_I \end{pmatrix}. \tag{122}$$

Then the idempotency relation for the G_χ are defined by the matrix multiplication rule $G = GG$.

The mass interaction can be written in G_χ form as follows:

	G_0	G_1	G_2
L to R	$+1/3$	$+\zeta^{+8}/3$	$+\zeta^{-8}/3$
R to L	$-1/3$	$-\zeta^{+8}/3$	$-\zeta^{-8}/3$

(123)

In matrix form, this is:

$$\Delta G = \pm\frac{1}{3}\begin{pmatrix} 1 & \zeta^{+8} & \zeta^{-8} \\ \zeta^{-8} & 1 & \zeta^{+8} \\ \zeta^{+8} & \zeta^{-8} & 1 \end{pmatrix}. \tag{124}$$

The above is a primitive idempotent circulant matrix.

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[6] The computation is by Java applet. Send an email if you would like a copy of the program.

5.5. The Generations

The preons of the previous section give the 32 particles of a single generation. The elementary particles include three generations, and to model them, we have to model how three preons combine to form a single fermion. Since the leptons are composed of preons identical except for precolor, we will concentrate on this problem.

One of the most useful facts about nature is that the mathematical objects that are convenient for describing composite particles turn out to also be useful in describing their component parts. Using this as an analogy, we look for a way of representing a composite density matrix from three density matrices.

Following a rough analogy from spinors, where one obtains a density matrix from the product of a ket with a bra:

$$\begin{pmatrix} \rho_a \\ \rho_b \\ \rho_c \end{pmatrix} \begin{pmatrix} \rho_a & \rho_b & \rho_c \end{pmatrix} = \begin{pmatrix} \rho_a & \rho_a\rho_b & \rho_a\rho_c \\ \rho_b\rho_a & \rho_b & \rho_b\rho_c \\ \rho_c\rho_a & \rho_c\rho_b & \rho_c \end{pmatrix}, \tag{125}$$

our model for the composite particle will be a 3×3 matrix with components given by products of primitive idempotents. Each of the $3 \times 3 = 9$ elements can be multiplied by an arbitrary complex constant a_{mn}. For example:

$$\rho_{\bar{e}R} = \begin{pmatrix} a_{11}\rho_{\bar{e}1} & a_{12}\rho_{\bar{e}1}\rho_{\bar{e}2} & a_{13}\rho_{\bar{e}1}\rho_{\bar{e}3} \\ a_{21}\rho_{\bar{e}2}\rho_{\bar{e}1} & a_{22}\rho_{\bar{e}2} & a_{23}\rho_{\bar{e}2}\rho_{\bar{e}3} \\ a_{31}\rho_{\bar{e}3}\rho_{\bar{e}1} & a_{32}\rho_{\bar{e}3}\rho_{\bar{e}2} & a_{33}\rho_{\bar{e}3} \end{pmatrix}. \tag{126}$$

There are therefore 9 complex degrees of freedom in this object.

The hope is that the above matrix, despite being a complicated composite object, will nevertheless have an interpretation as a simple density matrix. In our geometric language, density matrices are represented as geometric objects and the pure states are represented by primitive idempotents.

The way that one obtains primitive idempotents in a Clifford algebra is by multiplying (non primitive) commuting projection operators together. We therefore make the assumption that a preon can be factored into two commuting projection operators, one for precolor, and the other for all other quantum numbers. For example:

$$\rho_{\bar{e}R1} = P_1 \, P_{\bar{e}R}. \tag{127}$$

Thus the right-handed electron density matrix can be factored as follows:

$$\rho_{\bar{e}R} = P_{\bar{e}R} \begin{pmatrix} a_{11}P_1 & a_{12}P_1P_2 & a_{13}P_1P_3 \\ a_{21}P_2P_1 & a_{22}P_2 & a_{23}P_2P_3 \\ a_{31}P_3P_1 & a_{32}P_3P_2 & a_{33}P_3 \end{pmatrix}. \tag{128}$$

In the above, we take the matrix as indicating the particle's generation, while $P_{\bar{e}R}$ defines the particle's weak hypercharge and weak isospin quantum numbers.

In quantum mechanics, linear superposition is limited in that it is only allowed between states that are in the same superselection sector. In Eq. (36), $P_{\bar{e}R}$ defines the superselection sector, while the matrix is an array of projection operators that are shared between generations (and possibly all the fermions).

Since Eq. (36) is written with products such as $P_1\ P_2$, we cannot have that these precolor projection operators mutually annihilate. We will suppose that they come from a subalgebra isomorphic to the Pauli algebra. Thus P_n are to be represented by pure state density matrices, and we can fully characterize their multiplication by the angles between them, θ_{12}, θ_{13}, and θ_{23}. Furthermore, we can compute their products using the techniques of Subsection (1.5).

The simplest assumption for the θ_{mn} angles is that they are all $\pi/2$. Classically, this would mean that their waves do not interfere. As a mnemonic for calculational purposes, we will replace our color designations 1, 2, and 3 with x, y, and z and use the primitive idempotent notation appropriate for the Pauli algebra, ρ_x, ρ_y, and ρ_z.

It is useful to have a table of products of projection operators. A full table would be 9×9, but for our computations we need only 9×3 products. See Table (2).

TABLE 2. Products of projection operators for Pauli algebra spin in the x, y, and z directions. Similar for cyclic permutations of x, y, and z.

	ρ_x	$\rho_x\rho_y$	$\rho_x\rho_z$
ρ_x	ρ_x	$\rho_x\rho_y$	$\rho_x\rho_z$
$\rho_y\rho_x$	$\rho_y\rho_x$	$0.5\rho_y$	$0.5e^{-i\pi/4}\rho_y\rho_z$
$\rho_z\rho_x$	$\rho_z\rho_x$	$0.5e^{+i\pi/4}\rho_z\rho_y$	$0.5\rho_z$

We can now find what values of a_{mn} satisfy the idempotency equation $\rho^2_{\bar{e}R} = \rho_{\bar{e}R}$. Using Table (2) applied to Eq. (128), we find nine equations in nine unknowns:

$$\begin{aligned} a_{11} &= a_{11}^2 + 0.5a_{12}a_{21} + 0.5a_{13}a_{31}, \\ a_{22} &= a_{22}^2 + 0.5a_{21}a_{12} + 0.5a_{23}a_{32}, \\ a_{33} &= a_{33}^2 + 0.5a_{31}a_{13} + 0.5a_{32}a_{23}, \end{aligned} \tag{129}$$

$$\begin{aligned} a_{12} &= a_{11}a_{12} + a_{12}a_{22} + 0.5e^{-i\pi/4}a_{13}a_{32}, \\ a_{23} &= a_{22}a_{23} + a_{23}a_{33} + 0.5e^{-i\pi/4}a_{21}a_{13}, \\ a_{31} &= a_{33}a_{31} + a_{31}a_{11} + 0.5e^{-i\pi/4}a_{32}a_{21}, \end{aligned} \tag{130}$$

$$\begin{aligned} a_{13} &= a_{11}a_{13} + a_{13}a_{33} + 0.5e^{+i\pi/4}a_{12}a_{23}, \\ a_{21} &= a_{22}a_{21} + a_{21}a_{11} + 0.5e^{+i\pi/4}a_{23}a_{31}, \\ a_{32} &= a_{33}a_{32} + a_{32}a_{22} + 0.5e^{+i\pi/4}a_{31}a_{12}, \end{aligned} \tag{131}$$

Two solutions are obvious, the 3×3 matrices 0 and 1.

There are six other solutions, but three of them can be written as sums of the other three and so are not primitive. This leaves just three solutions that are primitive idempo-

tents and so suitable for treating as pure states. They can be written as:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 & \sqrt{2}e^{+\hat{i}\varepsilon_n} & \sqrt{2}e^{-\hat{i}\varepsilon_n} \\ \sqrt{2}e^{-\hat{i}\varepsilon_n} & 1 & \sqrt{2}e^{+\hat{i}\varepsilon_n} \\ \sqrt{2}e^{+\hat{i}\varepsilon_n} & \sqrt{2}e^{-\hat{i}\varepsilon_n} & 1 \end{pmatrix}, \qquad (132)$$
$$\varepsilon_n = 2n\pi/3 + \delta, \quad \delta = \pi/12$$

where n runs from 1 to 3. In the above we have split the angular contribution into two parts, $2n\pi/3$ and $\pi/12$. The reason for keeping these separate is that the $\pi/12$ arises from the fact that $\rho_x\rho_y\rho_z = \sqrt{2}\rho_x\rho_z e^{3\hat{i}\pi/12}$. If we had defined $\hat{i} = \hat{z}\hat{y}\hat{x}$ instead of $\hat{x}\hat{y}\hat{z}$, we would have had $\delta = -\pi/12$. And in later extensions, we will find that this angle can take other values. On the other hand, the $2n\pi/3$ comes from the complex matrix primitive idempotents alone.

As a hint to the algebra, note that the nine equations in nine unknowns can be transformed to the equations for matrix multiplication if one scales the off diagonal a_{nm} by multiplying them by $\sqrt{2}\exp(\pm i\pi/12)$. This reduces the problem to that of finding the primitive idempotents among the 3×3 complex matrices.[7]

The three solutions mutually annihilate. That is, the 3×3 matrix solutions multiply to zero when $n \neq m$. This is exactly what one expects of density operators for different states, so we associate these three solutions with the three different particles of the three fermion generations. The choice of $n = 1, 2, 3$ instead of $n = 0, 1, 2$ is so that our model will match the three generations when we calculate masses later in this paper. That is, we will associate the electron generation with $n = 1$, the muon with $n = 2$, and the tau with $n = 3$.

5.6. Interpretation of the primitive idempotent 3×3 matrices.

There were a total of eight solutions to the idempotency equation for 3×3 matrices. The non primitive solutions are then interpreted as situations where the number of particles present is different from one. The three solutions add to the unit matrix, so the unit matrix would be the representation of a right handed electron, muon and tau all being simultaneously present.

In addition to being thought of as states themselves, density operators also act as operators on the state vectors. A density operator takes an initial state $|I\rangle$ to a final state $|F\rangle$, for example:

$$|F\rangle = \rho\ |I\rangle. \qquad (133)$$

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[7] Note that these matrix solutions are "circulant". Circulant complex 3×3 matrices have already seen use in modeling the mixing angles of the leptons before, see [11] and references.

For the case of the 3×3 matrices, the initial and final states are evidently three preons, for example:

$$|I\rangle = \begin{pmatrix} \rho_x \\ \rho_y \\ \rho_z \end{pmatrix}. \tag{134}$$

The nine coefficients a_{mn} are therefore interpreted as the amplitudes for transitions between the preons. The requirement of idempotency is interpreted as the requirement that the situation be stable – the elementary particles are the things that propagate without changing.

The three diagonal elements, for example $a_{11}\ \rho_x$ correspond to cases where the preon final states are identical to the initial state. In the language of Green's functions, these are the bare preon propagators. The off diagonal elements, for example $a_{12}\ \rho_x\ \rho_y$ are for situations where the preons change state, and therefore correspond to propagators for interactions. As interactions, these off diagonal elements are somewhat unsatisfying in that they do not hook up to a boson that mediates the force that causes this change.

The form of Eq. (128) is such that when one squares the matrix, the individual products are always formed of like objects. For example, one finds products like $(P_a P_1)(P_1 P_b)$ where a and b are arbitrary, never products like $(P_a P_1)(P_2 P_b)$ where the two inner states are different. We interpret this to mean that in this model, the transitions between states are entirely contained in the matrix itself; between interactions of the sort modeled by the matrix, the particles propagate unchanged. In fact, if two products of states are compatible, for example $(P_a P_1)$ and $(P_1 P_b)$, then they will appear in the matrix multiplication exactly once. The matrix itself is therefore just a convenient way of organizing products of projection operators so as to arrange all possible products of compatible operations.

This paper is devoted to a quantum information type of analysis of the elementary particles, and in such an analysis it is not uncommon to leave off the gauge bosons. In addition, since we are analyzing structure inside of a point particle, we can also suppose that the gauge bosons exchanged are all created and annihilated at that same point and so need no propagators and can be safely treated as an external force field. Since the sum of the absorptions to and from the external force field match (that is, the initial state is the same as the final state), this force field fully cancels at each point in spacetime and can be ignored. In addition, since the number of particles in the initial and final states are the same, we can use quantum mechanics instead of quantum field theory to analyze the problem.

So in this treatment we can think of the three generations as arising from a difference not in the valence preons, but instead in the interactions between preons. The reason there are exactly three generations is then due to there being three primitive idempotent solutions to the 3×3 complex matrix idempotency equation. The nine coefficients roughly give the amount of activity (straight propagation or conversion from one sort preon to another) that is happening at a point in spacetime.

In general, things that are linear in spinor form become non linear in density operator form and vice versa. The idempotency relation $\rho^2 = \rho$ is manageable in the density operator language, but in the spinor language it can become messy. In the present case, the generations show up as three different levels of interaction that are consistent with a pure state. In the spinor language this would be three ways of dressing a propagator

under a highly nonlinear force. Solving the problem even approximately would be difficult but using the nonlinear density operator language we have solved it exactly.

5.7. The Geometric Trace

In the matrix representation of density operators, the trace is a very important function. A slightly different object, which we will find useful in similar ways, is the scalar part. For a real Clifford algebra, the scalar part will always be real, and this distinguishes it from the trace. For a complex Clifford algebra, the scalar part can be complex and the similarity is increased. We will follow the Clifford algebra "blade" notation and designate the scalar part of a Clifford algebra element Q as $\langle Q\rangle_0$. The notation and use is reminiscent of the spinor notation for the average of an operator.

For example, let the Pauli algebra be considered as a complex Clifford algebra. An arbitrary element of the algebra can be defined by four complex numbers:

$$Q = a_1 + a_x\hat{x} + a_y\hat{y} + a_z\hat{z} \tag{135}$$

and thus $\langle Q\rangle_0 = a_1$. Translating this into Pauli matrices, we have:

$$Q = \begin{pmatrix} a_1 + a_z & a_x - ia_y \\ a_x + ia_y & a_1 - a_z \end{pmatrix} \tag{136}$$

and so $\mathrm{tr}(Q) = 2a_1$. From this we see that the trace is twice the scalar part for the Pauli matrices. More generally, the trace will be N times the scalar part when the representation uses $N \times N$ matrices.

The nine elements of the matrix Eq. (128) each correspond to a Feynman diagram, a way of describing a part of a physical interaction. As such, the matrix organization is simply a convenient way of using matrix multiplication to keep track of which final states are suitable to be used as initial states. In particular, the three generation solutions shown in Eq. (132) are distinguished only by their off diagonal elements. Consequently, if we defined the trace as a sum of the diagonal elements we would miss the differences between the generations.

A natural operator to define on an object that contains nine operators is the sum of all the operators, and a natural function to define on this is the trace of the sum:

$$\begin{pmatrix} a_{11}P_1 & a_{12}P_1P_2 & a_{13}P_1P_3 \\ a_{21}P_2P_1 & a_{22}P_2 & a_{23}P_2P_3 \\ a_{31}P_3P_1 & a_{32}P_3P_2 & a_{33}P_3 \end{pmatrix} \rightarrow \mathrm{tr}\left(\sum_{j,k=1}^{3} a_{jk}P_jP_k\right) \tag{137}$$

Alternatively, one could take the nine individual traces and add them together. This is the definition of the scalar part that we will use for matrices of primitive idempotents.

If one were to follow the usual definition of the trace for matrices, one might expect that the trace for the 3×3 matrix of projection operators should include only the diagonal terms. If we were to define our function this way, we would be ignoring the off diagonal terms. But the three generations differ only in these off diagonal terms, so if we wish

an operator to distinguish between the generations, we must take into account the off diagonal terms.

Let us set $P_1 = (1+\hat{x})/2$, $P_1 = (1+\hat{y})/2$, and $P_1 = (1+\hat{z})/2$, and define the solution for three generations given in Eq. (132) as the "Pauli$_n$" solutions. We can calculate the scalar part of this by summing up the nine scalar parts that it is composed of:

$$\begin{aligned} \langle \text{Pauli}_n \rangle_0 &= \sum_j = 1^3 \sum_k = 1^3 \langle a_{jk} P_j \, P_k \rangle_0, \\ &= \sum_j = 1^3 \sum_k = 1^3 a_{jk} \langle a_{jk} P_j \, P_k \rangle_0, \end{aligned} \tag{138}$$

where in the second line we have supposed that the imaginary parts will cancel, a supposition that will turn out to be true. If $j = k$, then $P_j \, P_k = P_j$, and $\langle P_j \rangle_0 = 1/2$. If $j \neq k$, then the projection operators will multiply, for example $(1+\hat{x})/2(1+\hat{y})/2 = (1+\hat{x}+\hat{y}+\hat{x}\hat{y})/4$, and the scalar part will be $1/4$. The three diagonal elements all have scalar values of $(1/3) \times 1/2 = 1/6$. The six off diagonal elements have values of $(1/3)\sqrt{2}e^{\pm i\varepsilon_n} \times 1/4$ or $(1/12)\sqrt{2}\cos(\varepsilon_n)$. This gives a total for the scalar value of:

$$\begin{aligned} \langle \text{Pauli}_n \rangle_0 &= 3\tfrac{1}{6} + 6\tfrac{\sqrt{2}}{12}\cos(\varepsilon_n), \\ &= 0.5(1+\sqrt{2}\cos(\varepsilon_n)), \\ &= 0.5(1+\sqrt{2}\cos(2n\pi/3+\delta)), \end{aligned} \tag{139}$$

where $\delta = \pm\pi/12$.

6. THE KOIDE MASS FORMULA

The formula for $\langle \text{Pauli}_n \rangle_0$ given in Eq. (139) gives three scalar numbers that are to be associated with the three generations. This suggests that they should have something to do with the masses, and so we should compare the formula with the masses of the elementary fermions.

In 1982, Yoshio Koide [12, 13], discovered a formula relating the masses of the charged leptons:

$$\frac{(\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau})^2}{m_e+m_\mu+m_\tau} = \frac{3}{2}. \tag{140}$$

The formula was used to predict the value of the tau long before accurate measurements of it had been obtained. Over the years, the masses of the charged leptons have been measured to ever greater experimental accuracy and the Koide formula still holds. At the present time, it is quite close to the centers of the error bars.[14]

In the context of the Higgs, the accuracy of the Koide formula is mysterious for several reasons.[15] The most severe difficulty is that the formula is exact at low energies. The Higgs model is based on perturbation theory that should give exact results at high energies, to see a simple formula arise at low energies is not expected. Furthermore, renormalization scales the charged leptons differently and so would destroy the simplicity of the Koide formula.[16]

The present paper describes a non perturbational method of relating the density operator models of bound states to the density operators of their unbound components.

As such it provides a tool for exploring the relationships between generations that does not require perturbation theory or renormalization. The difficulties of the Koide formula for standard particle physics do not apply here.

To compare the Koide formula with Eq. (139), we need to rewrite the Koide formula to a form which gives the masses of the charged leptons. To do this, let us suppose that the square roots of the lepton masses form three eigenvalues of a complex 3×3 circulant matrix:

$$M = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}. \tag{141}$$

Such a matrix will have real eigenvalues only if it is Hermitian, so A is real, and $B^* = C$. We rewrite the matrix by pulling out a scale factor μ and writing $B = \eta e^{i\alpha}$ to give:

$$M = \mu \begin{pmatrix} 1 & \eta e^{+i\alpha} & \eta e^{-i\alpha} \\ \eta e^{-i\alpha} & 1 & \eta e^{+i\alpha} \\ \eta e^{+i\alpha} & \eta e^{-i\alpha} & 1 \end{pmatrix}. \tag{142}$$

The Koide formula deals with the sums of square roots of masses and sums of masses. The above matrix is to have as its eigenvalues the square roots of the masses of the charged leptons, so

$$\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau} = 3\mu, \tag{143}$$

as the sum of the eigenvalues of a matrix is given by its trace.

When a matrix is squared, its eigenvalues are also squared, so to get the sum of the masses of the charged leptons we square M. This gives:

$$m_e + m_\mu + m_\tau = \text{ tr}(M^2) = 3\mu^2(1 + 2\eta^2). \tag{144}$$

To obtain the Koide formula we therefore require that

$$(3\mu)^2 = \frac{3}{2}(3\mu^2(1 + 2\eta^2)) \tag{145}$$

and therefore

$$\eta = \sqrt{1/2}. \tag{146}$$

The eigenvectors of a circulant matrix are three vectors familiar from the solving of the idempotency equation for 3×3 complex circulant matrices:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ e^{+2i\pi/3} \\ e^{-2i\pi/3} \end{pmatrix}, \begin{pmatrix} 1 \\ e^{-2i\pi/3} \\ e^{+2i\pi/3} \end{pmatrix}. \tag{147}$$

Applying these to M, the three square roots of masses are therefore:

$$sqrtm_n = \mu(1 + \sqrt{2}\cos(2n\pi/3 + \alpha). \tag{148}$$

The value of 3α from recent PDG[17] data works out to be:[18]

$$\alpha = 0.22222204717(48). \tag{149}$$

This number is rather close to $2/3$, but no one has yet come up with a reason why this should be so.[8] Comparing this with Eq. (139), we see that the Koide formula is similar in form, but instead of the mathematical value $\pi/12 = 0.2617993878(1)$, we need a slightly smaller value.

6.1. The Angle δ

If we loosen the requirement that P_1, P_2, and P_3 are at right angles to each other, we can arrange for the angle δ to be other than $\pm\pi/12$, however this will change the ratio of the off diagonal elements of Eq. (139) to the diagonal elements. Suppose that the angles between P_1, P_2, and P_3 are all equal to α (the side angle of the equilateral spherical triangle) so that transitions between these states have amplitudes of $(1+\cos(\alpha))/2$. By the spherical law of cosines, we have

$$\cos(\beta) = \frac{\cos(\alpha)}{1+\cos(\alpha)}, \tag{150}$$

where β is the corner angle of the equilateral spherical triangle. By the law of spherical excess, the surface area of the spherical triangle is $3\beta - \pi$. The transition amplitude a and phase b are therefore given in terms of α by:

$$\begin{array}{rcl} a^2 & = & (1+\cos(\alpha))/2, \\ 2b & = & 3\cos^{-1}(\cos(\alpha)/(1+\cos(\alpha))) - \pi. \end{array} \tag{151}$$

Therefore the equivalent 3×3 matrix will have the off diagonal elements multiplied by $\sqrt{2/(1+\cos(\alpha))}e^{\pm i(\arccos(\cos(\alpha)/(1+\cos(\alpha)))-\pi/3)/4}$. In particular, it will be impossible to obtain the peculiar angle $0.22222204717(48)$ of Eq. (149) without losing the $\sqrt{2}$ of Eq. (148) and Eq. (139).

Another way of attempting to obtain this relation is to replace the usual primitive idempotents of the Pauli algebra with non Hermitian primitive idempotents. The Hermitian primitive idempotents of the Pauli algebra are characterized by real unit vectors. Given a real unit vector $\vec{u} = (u_x, u_y, u_z)$, the corresponding Hermitian primitive idempotent is $(1+\vec{u}\cdot\vec{\sigma})/2$.

To generalize this to the non Hermitian primitive idempotents, one allows the unit vector to range over the reals. That is, one allows u_x, u_y, and u_z to be complex, subject to the restriction that $u_x^2 + u_y^2 + u_z^2 = 1$. The solutions to this equation are characterized

[8] In tribute to this problem, Marni Sheppeard, Michael Ruis, and the author refer to this value as "that damned number". In fact, one can rewrite the formula using 3α to obtain a constant that begins with the two copies of the biblically significant number 666.

by two perpendicular real unit vectors $\vec{v}$ and $\vec{w}$ and an angle κ. The general solution for the complex vector $\vec{u}$ is then:

$$\vec{u} = \sec(\kappa)\vec{v} + i\tan(\kappa)\vec{w} \tag{152}$$

and the general non Hermitian primitive idempotent is

$$\rho_u = (1 + \vec{u}\cdot\vec{\sigma})/2 = (1 + \sec(\kappa)\vec{v}\cdot\vec{\sigma} + i\tan(\kappa)\vec{w}\cdot\vec{\sigma})/2. \tag{153}$$

With this change, one can obtain the $\sqrt{2}$ for more general situations than with the usual Hermitian primitive idempotents, but one still ends up with $\alpha = \pi/12$.[9] As an aside, any non Hermitian primitive idempotent of the Pauli algebra can be written in only one way as a real multiple of the product of two distinct Hermitian primitive idempotents.

The angle $\alpha = 0.22222204717(48)$ cannot be achieved in the Pauli algebra at the same time that the $\sqrt{2}$s appear off diagonal, however there is a way that it can be achieved in a more complicated algebra. To achieve this, we will begin by writing down two 3×3 primitive idempotent matrix solutions, one which gives $\delta = +\pi/12$, the other which gives $\delta = -\pi/12$. These two solutions correspond to two different particles. To obtain a solution compatible with the observed charged lepton masses, we will mix these two solutions.

The Pauli algebra can give only one result for δ, so we will now approach the problem from the Dirac algebra. The basis vectors are $\hat{x}$, $\hat{y}$, $\hat{z}$, and $\hat{t}$. We need to choose three elements that square to $+1$ in order to define the projection operators P_1, P_2, and P_3. Since our theory is directed at chiral wave states, and since the chiral wave states are massless and travel at speed c, a natural choice for these three elements are the velocity operators, $\hat{x}\hat{t}$, $\hat{y}\hat{t}$, and $\hat{z}\hat{t}$. Then we have:

$$\begin{array}{rcl} P_1 &=& (1+\hat{x}\hat{t})/2, \\ P_2 &=& (1+\hat{y}\hat{t})/2, \\ P_3 &=& (1+\hat{z}\hat{t})/2, \end{array} \tag{154}$$

Our choice of these three projection operators defines the "imaginary unit" $\hat{i}$ in our mass formula:

$$\begin{array}{rcl} \hat{i} &=& (\hat{x}\hat{t})(\hat{y}\hat{t})(\hat{z}\hat{t}), \\ &=& \hat{x}\hat{y}\hat{z}\hat{t}. \end{array} \tag{155}$$

This "$\hat{i}$" is not a scalar that commutes with everything in the algebra, but for our situation we don't need it to be. The scalar part of the 3×3 matrix takes only the real parts of the products of projection operators, so $\hat{i}$ does not appear in our formula at all. In particular, note that $\hat{i} \neq i$.

The Dirac algebra requires two quantum numbers to define the states. The operators for the first quantum numbers for the three states we're dealing with are to be $\hat{x}\hat{t}$, $\hat{x}\hat{t}$, and $\hat{x}\hat{t}$. There is only one operator possible that commutes with all of these and it is $\hat{x}\hat{y}\hat{z}\hat{t}$.

[9] Numerical search by computer.

This squares to -1, so we must first multiply it by i. We will think of this operator as a sort of charge:

$$Q = i\hat{x}\hat{y}\hat{z}\hat{t} = i\hat{i}. \tag{156}$$

The projection operators for $Q = \pm 1$ are:

$$P_{\pm} = (1 \pm Q)/2 = (1 \pm \hat{x}\hat{y}\hat{z}\hat{t})/2. \tag{157}$$

and we will be dealing with the six states:

$$\begin{array}{rcl} \rho_{\pm 1} & = & (1+\hat{x}\hat{t})(1 \pm \hat{x}\hat{y}\hat{z}\hat{t})/4, \\ \rho_{\pm 2} & = & (1+\hat{x}\hat{t})(1 \pm \hat{y}\hat{y}\hat{z}\hat{t})/4, \\ \rho_{\pm 3} & = & (1+\hat{x}\hat{t})(1 \pm \hat{z}\hat{y}\hat{z}\hat{t})/4. \end{array} \tag{158}$$

Note that these six states are eigenvectors of Q with eigenvalues of ± 1. For example, $Q\rho_{\pm 1} = \pm 1 \rho_{\pm 1}$. Thus we have:

$$\hat{i}\rho_{\pm 1} = -iQ\rho_{\pm 1} = \mp i \rho_{\pm 1}. \tag{159}$$

Using this equation, we can remove the hats from the $\hat{i}$s in our mass formula. But in doing this, we get $-i$ or $+i$ depending on the quantum number for Q.

This is exactly the behavior we desire. States with a Q quantum number of -1 take a $+\pi/12$ when computing $P_1\, P_2\, P_3$, and states a Q quantum number of $+1$ take a $-\pi/12$. All we need to do is to define a mixed state.

6.2. Mixed States and δ

The mathematics of quantum mechanics allows any two spin $1/2$ fermion states to be combined by linear superposition. However, when the fermion states are to represent two physical objects, the superposition may or may not be physically observed. For example, the linear superposition of an electron with a neutrino is never observed in experiment. Two quantum objects may be put into linear superposition when they are in the same "superselection sector." To allow mixed states between states with $\delta = +\pi/12$ and states with $\delta = -\pi/12$, in standard quantum mechanics we need to have that the states to be mixed are in the same superselection sector.

Standard quantum mechanics requires superselection sectors because otherwise one could add together any two spinors and obtain a mixed state between them. In Subsection (2.1), we showed that in the density operator theory, we can obtain spinors by adding a vacuum state. This is the method that we will use to find a linear superposition of the $\delta = \pm\pi/12$ states. That is, we will look for a vacuum state ρ_0 and combine ρ_{+n} and ρ_{-n} by

$$|\alpha n\rangle = (\alpha\rho_{+n} + (1-\alpha)\rho_{-n})\, \rho_0, \tag{160}$$

where α is a complex constant, and n is $1, 2$, or 3. The resulting density operator is:

$$\rho_{\alpha n} = (\alpha\rho_{+n} + (1-\alpha)\rho_{-n})\, \rho_0 (\alpha^*\rho_{+n} + (1-\alpha^*)\rho_{-n}), \tag{161}$$

which is not necessarily normalized. Then, as α change from 0 to 1, $\rho_{\alpha 1}$ changes continuously from ρ_{-1} to ρ_{+1}. A mixed state with the desired δ can be found by trying various values of α.

But there is a problem with this scheme. If our two states are connected by a series of states that are not perpendicular, they will lose the $\sqrt{2}$ seen in Eq. (148) and Eq. (139). Thus we have to keep our orientation projection operators, P_1, P_2, and P_3 unaltered. We can do this by arranging for the vacuum to commute with the P_n and then making the linear superposition calculation in the other than P_n projection operators only. Unfortunately, in the Dirac algebra, the only non trivial root of unity that commutes with all the P_n is the same $i\hat{x}\hat{y}\hat{z}\hat{t}$, and so the only choices we have for the vacuum are $(1 \pm i\hat{x}\hat{y}\hat{z}\hat{t})$ and these will annihilate one or the other of the objects we are trying to find linear superposition between. The short way of saying this is that, in the Dirac algebra, the ρ_{-n} and ρ_{+n} are in different superselection sectors.

So we augment the Dirac algebra by adding one more basis vector, $\hat{s}$; this squares to $+1$, commutes with the P_n, and commutes with $(1 \pm i\hat{x}\hat{y}\hat{z}\hat{t})/2$, so now instead of six states $\rho_{\pm n}$, we have 12 states:

$$\rho_{\pm\pm n} = (1 \pm \hat{s})(1 \pm i\hat{x}\hat{y}\hat{z}\hat{t})(1 \pm \hat{n}\hat{t})/8, \tag{162}$$

where n is any of x, y, or z, and the two $\pm$ are separate; the first in $\rho_{\pm\pm n}$ applying to the first $\pm$ on the right hand side.

With the augmented algebra, the roots of unity that commute with all of the P_n include $\{i\hat{x}\hat{y}\hat{z}\hat{t},\ \hat{s},\ i\hat{x}\hat{y}\hat{z}\hat{s}\hat{t}\}$. If we choose any two of these, the third will come along automatically in the cross products, so again it appears that any choice we make will be identical.

7. CONCLUSION

We have shown that quantum spin$-1/2$ states have a natural geometric representation as density operators entirely in the spin operator algebra. This representation avoids the need to choose a matrix representation of the operators. The notation is simple, elegant, and concise, and it gives computations that are considerably easier than that of the traditional Pauli spinors. Furthermore, the method is written entirely in objects that have geometric significance. We have shown that the linear superposition of spinors is more completely understood in the density operator formalism than it is in the spinor formalism.

The existence of a geometric algebra representation of spin$-1/2$ quantum states suggests that researchers should consider geometrizing the standard model using the density operator formalism. Schwinger's measurement algebra is compatible with the density operators and provides a natural extension to algebras (and therefore to symmetries) beyond that of the Pauli and Dirac algebras. The absence of a vacuum in the Schwinger measurement algebra suggests that some of the problems in the standard model arise from excessive reliance on vacuum expectation values for providing particle masses.

Since qubits, in the density operator formalism, are primitive idempotents of the Clifford algebra $R(3,0)$, we compared the structure of the idempotents of more complicated Clifford algebras with the structure of the elementary fermions of the standard model.

We find that the structures match if we assume one hidden dimension, and a preon structure for the quarks and leptons. Applying density operator theory to that preon structure, we obtain a derivation for the Koide relation for the masses of the charged leptons and predict the masses of the neutrinos.

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