Feynman's Checkerboard, the Dirac Equation and Spin

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ABSTRACT

Spin is one of the fundamental observables in quantum mechanics lacking a satisfactory physical picture. Early attempts to explain the physical origins of spin are briefly discussed here. Feynman was able to demonstrate in one space dimension and one time dimension (1+1) the equivalence of a particular path integral with the one-dimensional Dirac equation, thereby presenting a simple process from which the analogue of spin in one dimension, helicity, naturally arose. This path integral became known as Feynman's Checkerboard or Chessboard, named for the appearance of possible paths on a spacetime diagram. Subsequent work by others attempted to generalize this to three space dimensions (3+1) to obtain the full Dirac equation. The remainder of this paper will concentrate on rederiving Feynman's results and describing the relation to the Dirac equation and, interestingly enough, the one-dimensional Ising model. This paper will then conclude with an original interpretation to the 1+1 dimensional checkerboard problem.

1. Introduction - Early Interpretations of Spin

In classical mechanics, the notion of spin is only a convenience of terminology used to represent aggregate sums of angular momenta. Goudsmit and Uhlenbeck (and separately Kronig) initially proposed a physical picture for the electron spin analogous to the classical spin, consisting of a rotating sphere the size of the classical electron radius, given by

$$r_c = \frac{e^2}{4\pi\epsilon_0 m_e c^2} \tag{1}$$

However, it was trivially shown that the velocity on the surface of this spinning electron model would necessarily exceed the speed of light in order to produce the quantized angular momentum of $\hbar/2$. The direct analogy to classical spin was abandoned, but the success of the model in explaining observable phenomena (e.g., Zeeman effect) led to the acceptance of a discretized angular momentum, intrinsic to the electron and all other particles (Ohanian 1986).

In 1928, Gordon proposed that the magnetic moment of electron spin could be constructed from a circular flow of charge in the electron's wave-field. Additionally, in 1939 Belinfante proposed that spin could be described by a circulating energy flow in the electron's wave-field. Both of these physical and quantum mechanical explanations for spin received little recognition, and Ohanian revisited the results of Belinfante and Gordon in 1986 (Ohanian 1986). Choosing a symmetric stress-energy tensor motivated by the corresponding symmetrization requirement in general relativity, computation of the angular momentum from the T^{k0} components leads to a conservation of angular momentum. With this symmetrized stress-energy tensor, the momentum density must then include both orbital and spin angular momentum components. In the Dirac field, the momentum density is then given by

$$\vec{G} = \frac{\hbar}{4i}\left(\psi^\dagger\vec{\nabla}\psi - \frac{1}{c}\psi^\dagger\vec{\alpha}\frac{\partial\psi}{\partial t}\right) + \text{h.c.} \tag{2}$$

where h.c. stands for hermitian conjugate and $\vec{\alpha}$ are the alpha matrices from the Dirac equation. Later in this paper, I will make use of the Weyl representation of the Dirac equation, where $\vec{\alpha}$ will be given explicitly by:

$$\vec{\alpha} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \tag{3}$$

where $\vec{\sigma}$ are the 2x2 Pauli matrices. Using the Dirac equation, equation (2) and some manipulation, Belinfante and Ohanian arrive at expressions for the momentum density and total conserved angular momentum of an arbitrary wave function:

$$\vec{G} = \frac{\hbar}{2i}[\psi^\dagger\vec{\nabla}\psi - (\vec{\nabla}\psi^\dagger)\psi] + \frac{\hbar}{4}\vec{\nabla}\times(\psi^\dagger\vec{\Sigma}\psi) \tag{4}$$

$$\vec{J} = \frac{\hbar}{2i}\int \vec{x}\times[\psi^\dagger\vec{\nabla}\psi - (\vec{\nabla}\psi^\dagger)\psi]d^3x + \frac{\hbar}{2}\int(\psi^\dagger\vec{\Sigma}\psi)d^3x \tag{5}$$

where $\Sigma_x = -i\alpha_y\alpha_z$, and Σ_y and Σ_z are obtained from Σ_x by cyclically permuting the indices.

For equation (4), the first term is the electron's translational momentum density, and Ohanian identifies the second term as the electron's rest frame energy flow. For equation (5), the first term is the orbital angular momentum, and the second term is identified by Ohanian as the spin. This latter identification is not suprising, as the Dirac equation includes spin in its formulation. What is interesting, however, is that Ohanian considered a finite wave-packet for ψ to represent a free electron. Instead of using a plane-wave solution to the

Dirac equation, this is motivated by the finite-extent of real particles. For example, take the Gaussian wave-packet:

$$\psi_{\text{trial}} = (\pi d^2)^{-3/4}\, e^{-(1/2)r^2/d^2}\; w^1(0) \tag{6}$$

Associating this with an electron at rest with spin up in the nonrelativistic limit, and substituting in for ψ into equation (4), the first term in (4) is zero, and the second term becomes:

$$\vec{G}_{\text{trial}} = \frac{\hbar}{4}\left(\frac{1}{\pi d^2}\right)^{3/2} \frac{e^{-r^2/d^2}}{d^2}(-2y\hat{x} + 2x\hat{y}) \tag{7}$$

Although a bit simplistic, ψ_{trial} generates a term from (4) that Ohanian and Belinfante identified as a circulating energy flow in the electron's wave-packet. This term then directly produces the associated spin term in equation (5), thus providing a physical picture for spin. Using a similar approach, Ohanian also reproduced Gordon's work in showing that a finite wave-packet also produces a circulating flow of charge in the electron's wave-packet, giving rise to a term matching the magnetic moment operator of spin,

$$\vec{m} = -(e/m)\gamma_0\vec{S} \tag{8}$$

where γ_0 is the familiar 4x4 gamma matrix in the Dirac equation formalism.

While the work of Belinfante and Gordon in the early part of the 20^{th} century received little recognition, it is interesting that a physical picture of spin arises naturally from the Dirac equation and a *finite extent* of particle wave function. The physical picture contains rotational motion intrinsic to the particle's wave-packet, but not internal to the particle itself. Rotational symmetry considerations were not taken into account, but the time-reversal symmetry holds. The energy flow reverses direction, and the spin 'up' ψ_{trial} becomes spin 'down' as expected.

2. Feynman's Checkerboard

2.1. Path Integrals

In the latter half of the 1940s, Feynman developed his path integral formulation of quantum mechanics, deriving propagators for particles by summing over all possible paths from (x_a, t_a) to (x_b, t_b). For instance, the non-relativistic free particle propagator in one dimension is given by:

$$K^{(\text{NR})}(x_b, t_b; x_a, t_a) = \lim_{N\to\infty}\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{\frac{1}{2}(N-1)} \int dx_1 \ldots dx_N\, e^{\frac{im\sum(x_{i+1}-x_i)^2}{2\epsilon}} \tag{9}$$

which can be evaluated explicitly to be

$$K^{(\mathrm{NR})}(x_b, t_b; x_a, t_a) = \sqrt{\frac{m}{2\pi i\hbar(t_b - t_a)}} \exp\left(\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right) \tag{10}$$

where in general,

$$K(\vec{x}_b, t_b; \vec{x}_a, t_a) = \langle \vec{x}_b | e^{\frac{-iH(t_b - t_a)}{\hbar}} | \vec{x}_a \rangle \tag{11}$$

and

$$\psi(\vec{x}_b, t_b) = \int d^3x_a K(\vec{x}_b, t_b; \vec{x}_a, t_a)\psi(\vec{x}_a, t_a) \tag{12}$$

The techniques and methods of path integrals were collected by Hibbs and published in 1965 by Feynman and Hibbs as *Quantum Mechancis and Path Integrals.* Problem 2-6 on pp34-36 of this textbook, lacking an obvious published precursor (all subsequent papers on the subject cite this problem directly), brought about a subtle connection to the Dirac equation and became known as Feynman's Checkerboard or Chessboard. I will now introduce the path integral and evaluate it exactly, following work by Jacobson and Schulman (1984).

2.2. The Checkerboard Path Integral

Feynman proposed a relativistic random walk process in 1+1 dimensions depicted in Figure 1. The slope of the segments is constant in magnitude and differs only in sign from segment to segment. I will be using units where $\hbar = c = 1$. Feynman noted that the propagator for this motion is given by (Feynman and Hibbs 1965):

$$K_{\beta\alpha}(x_b, t_b; x_a, t_a) = \lim_{N\to\infty} \sum_{R\geq 0} \Phi_{\beta\alpha}(R)(i\epsilon m)^R \tag{13}$$

where $\epsilon = (t_b - t_a)/N$ is the length of each step (ie, $c\,\epsilon$), α and β take the values of 'right' and 'left' and $\Phi_{\beta\alpha}(R)$ is the number of paths with exactly N steps that start at x_a and in direction α and end at x_b in the direction β and switch direction R times. As noted by Jacobson and Schulman (1984), "the space and time steps are of the *same* size in the sense that they scale the same way with N."

At this point, I make note of the following caveat - the relativistic particle in this problem is both massive and moving at the speed of light. These requirements, respectively, are necessitated in order for the particle to switch directions and for the space and time steps to scale the same with N. While this presents an apparent violation of special relativity, the kernel is only evaluated for net subluminal motion,

$$|x_b - x_a| < c(t_b - t_a) \tag{14}$$

This apparent discrepancy is explored in the latter part of this paper. Also, I note that sometimes the equivalence of random-walking light to the Dirac equation is made, where the number of reversals is Lorentz invariant. However, this is misleading as the particles are necessarily massive as already stated. I turn now to evaluating K_{-+}.

2.3. Derivation of Propagator

Consider a path with R bends that contributes to the sum for K_{-+}, which leaves moving right and arrives moving left. It makes $1 + (R-1)/2$ turns to the left and $(R-1)/2$ turns to the right, where the last turn is to the left and R is odd for K_{-+}. Defining P to be the number of steps to the right and Q to the left, such that $P + Q = N$, I have for the total number of paths from x_a to x_b for K_{-+}:

$$\Phi_{-+}(R) = \begin{pmatrix} P-1 \\ \frac{1}{2}(R-1) \end{pmatrix} \begin{pmatrix} Q-1 \\ \frac{1}{2}(R-1) \end{pmatrix} \tag{15}$$

Then, in the limit of $N \to \infty$,

$$\lim_{N\to\infty} \Phi_{-+}(R) = \frac{(PQ)^{\frac{1}{2}(R-1)}}{[(\frac{1}{2}(R-1))!]^2} \tag{16}$$

and I have

$$K_{-+}(x_b, t_b; x_a, t_a) = \sum_{R\geq 0,\ \text{R odd}} \frac{(PQ)^{\frac{1}{2}(R-1)}}{[(\frac{1}{2}(R-1))!]^2} (i\epsilon m)^R \tag{17}$$

Letting $M = (P - Q)$,

$$PQ = \frac{1}{4}(N+M)(N-M) = \left(\frac{N}{2\gamma}\right)^2 \tag{18}$$

where $\gamma = (1 - v^2)^{-1/2}$ and $v^2 = M^2/N^2 = (x_b - x_a)^2/(t_b - t_a)^2$. Using $\epsilon = (t_b - t_a)/N$, equation (17) becomes

$$K_{-+}(x_b, t_b; x_a, t_a) = \frac{2\gamma}{N} \sum_{R\geq 0,\ \text{R odd}} \frac{\left(\frac{im(t_b-t_a)}{2\gamma}\right)^R}{[(\frac{1}{2}(R-1))!]^2} \tag{19}$$

Letting $z = m(t_b - t_a)/\gamma$, replacing N with $(t_b - t_a)/\epsilon$, and letting $R = 2k+1$, equation (19) becomes

$$K_{-+}(x_b, t_b; x_a, t_a) = i\epsilon m \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{(k!)^2} = i\epsilon m J_0(z) \tag{20}$$

where J_0 is the zeroth Bessel function of the first kind. According to Jacobson and Schulman (1984), I divide by 2ϵ to get the continuum limit as $\epsilon \to 0$ since K_{-+} vanishes at every other lattice point. Noting,

$$z = m(t_b - t_a)/\gamma = m(t_b - t_a)\sqrt{1 - v^2} = m\sqrt{(t_b - t_a)^2 - (x_b - x_a)^2} \equiv m\tau \tag{21}$$

I finally arrive at the exact continuum propagator $K_{-+}(x_b, t_b; x_a, t_a) = i\frac{m}{2}J_0(m\tau)$ in agreement with Jacobson and Schulman (1984). Similar calculations can be done for the components of $K_{\alpha\beta}$ with different factors of Φ. Note, Φ_{++} can be inferred from Figure 1 to be:

$$\Phi_{++} = \begin{pmatrix} P-2 \\ \frac{1}{2}R \end{pmatrix} \begin{pmatrix} Q-1 \\ \frac{1}{2}R - 1 \end{pmatrix} \tag{22}$$

and similarly,

$$\Phi_{--} = \begin{pmatrix} P-1 \\ \frac{1}{2}R - 1 \end{pmatrix} \begin{pmatrix} Q-2 \\ \frac{1}{2}R \end{pmatrix} \tag{23}$$

Following the same procedure as before, Jacobson obtains all four components of the continuum propagator. Letting $x_b = x$, $x_a = 0$, $t_b = t$ and $t_a = 0$

$$K(x,t;0,0) = \frac{im}{2}\begin{pmatrix} \frac{i(t+x)}{\tau}J_1(m\tau) & J_0(m\tau) \\ J_0(m\tau) & \frac{i(t-x)}{\tau}J_1(m\tau) \end{pmatrix} \tag{24}$$

where τ becomes $\tau = \sqrt{t^2 - x^2}$ with $|x| < t$. It should be noted that Kull and Treumann (1999) obtained the same result as equation (24) without an overall multiplicative factor of $\frac{im}{2}$.

2.4. Connection with the Dirac Equation and the One-Dimensional Ising Model

The connection of the above result with the one-dimensional Dirac equation is not obvious, where the one-dimensional Dirac equation is given by

$$i\partial_t\Psi = -i\sigma_z\partial_x\Psi - m\sigma_x\Psi \tag{25}$$

where σ_z and σ_x are 2x2 Pauli matrices and Ψ is a two-component Dirac spinor, and I still have $\hbar = c = 1$. Jacobson and Schulman (1984) and Kull and Treumann (1999) present different interpretations of (24). I will consider Kull's first, as it is more direct. Choosing two Dirac spinors given by

$$\Psi_1 = \begin{pmatrix} K_{++} \\ K_{+-} \end{pmatrix}, \; \Psi_2 = \begin{pmatrix} K_{-+} \\ K_{--} \end{pmatrix} \tag{26}$$

where from before we have $K_{+-} = K_{-+}$ and using either representation of $K_{\alpha\beta}$ (with or without the factor of $\frac{im}{2}$, as it will drop out in the end), one can easily verify Ψ_1 and Ψ_2 are two independent, exact solutions of the Dirac equation given in (25). Thus, Kull concludes that Feynman's Checkerboard yields solutions to the 1+1 Dirac equation.

Jacobson and Schulman (1984) presents a different approach, going back to equation (13). Redefining the sum with N variables $\mu_i = \pm 1$, representing a spatial step to the right or left on the i^{th} time step that satisfy $M = \sum_i \mu_i = (x_b - x_a)/\epsilon$, equation (13) becomes:

$$K_{\beta\alpha} = \sum_{\mu_2=\pm 1} \dots \sum_{\mu_{N-1}=\pm 1} (i\epsilon m)^R \tag{27}$$

Furthermore, recognizing

$$R = \frac{1}{2}\sum_{i=1}^{N-1}(1 - \mu_i\mu_{i+1}) \tag{28}$$

and letting $\nu = -\frac{1}{2}log(i\epsilon m)$, Jacobson and Schulman (1984) made the identification of (27) with the partition function of the one-dimensional Ising model, with (coupling constant/temperature) $= \nu$, and the condition that $M = \sum_i \mu_i$ being interpreted as evaluating the paritition function at a fixed magnetization as opposed to a fixed external field. Rewriting this contraint on the μ_i's as a Kronecker delta:

$$\delta\left(M, \sum \mu_i\right) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta(M - \sum \mu_i)} \tag{29}$$

we obtain the usual form of the one-dimensional Ising Model for equation (27), with unconstrained sums on the μ_i's,

$$K_{\beta\alpha} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{iM\theta} \sum_{\mu_2 \dots \mu_{N-1}} exp\left(\nu \sum_{i=1}^{N-1} \mu_i\mu_{i+1} - i\theta \sum_{i=1}^{N} \mu_i - (N-1)\nu\right) \tag{30}$$

Using techniques similar to those used to solve the one-dimensional Ising model, by defining the transfer matrix

$$L(\mu, \mu') = exp\left(\nu\mu\mu' - \frac{1}{2}i\theta(\mu + \mu') - \nu\right) \tag{31}$$

equation (30) becomes

$$K_{\beta\alpha} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{iM\theta} \left(L^{N-1}\right)_{\beta\alpha} e^{-\frac{1}{2}i\theta(\alpha+\beta)} \tag{32}$$

where α and β are $+1$ and -1 for 'right' and 'left' respectively in the last exponential in equation (32). Evaluating the eigenvalues and eigenvectors of L, and defining the projection operators for the eigenvectors, which can be written as (with some algebraic manipulation):

$$P_{\pm} = \frac{1}{2}\left(1 \pm \left(\sigma_x \frac{m}{\sqrt{m^2+p^2}} - \sigma_z \frac{p}{\sqrt{m^2+p^2}}\right)\right) \tag{33}$$

equation (32) becomes

$$K_{\beta\alpha} = \sum_{k=\pm 1} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{iM\theta} \left((P_k)_{\beta\alpha}\left(cos(\theta) + ik\sqrt{\epsilon^2 m^2 + sin^2(\theta)}\right)^{N-1}\right) e^{-\frac{1}{2}i\theta(\alpha+\beta)} \tag{34}$$

Finally, Jacobson and Schulman (1984) make the substitution $\theta = p\epsilon$. Taking the limit of ϵ to be small and thus dropping terms of order ϵ^2 and higher, equation (34) becomes

$$K_{\beta\alpha} = \sum_{k=\pm 1} \frac{\epsilon}{2\pi} \int_{-\pi/\epsilon}^{\pi/\epsilon} dp\, e^{ipx} \frac{1}{2}\left(1 + k\left(\sigma_x \frac{m}{\sqrt{m^2+p^2}} - \sigma_z \frac{p}{\sqrt{m^2+p^2}}\right)\right)_{\beta\alpha} \left(1 + ik\epsilon\sqrt{m^2+p^2}\right)^{N-1} \tag{35}$$

where $x = x_b - x_a$. Using $t = N/\epsilon$, summing over k, and using the limit of $(1+A/N)^N \rightarrow e^A$ for $N \rightarrow \infty$ for the last term in equation (35), equation (35) finally evaluates to:

$$K_{\beta\alpha}(x,t;0,0) = \frac{\epsilon}{2\pi} \int dp\, e^{ipx} \left(e^{it(m\sigma_x - p\sigma_z)}\right)_{\beta\alpha} \tag{36}$$

This is simply the Fourier transform of

$$e^{-iHt} \tag{37}$$

where H is the Hamiltonian for the one-dimensional Dirac equation given by equation (25). The factor of ϵ comes from the use of a discrete spatial lattice. The equivalence of equations (24) and (36) in the continuum limit is not readily apparent, but I assert without proof that this can be accomplished with equation (34) and the integral representations of the Bessel functions,

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{iz\, cos(\theta)} d\theta \tag{38}$$

and

$$J_1(z) = \frac{1}{2\pi i} \int_0^{2\pi} e^{iz\, cos(\theta) + i\theta} d\theta \tag{39}$$

Thus, I have shown the connection and equivalence between Feynman's checkerboard and the 1+1 Dirac equation, as well as a connection with the one-dimension Ising model. Feynman's path integral provides another physical process - the relativistic random walk - from which the one-dimensional analogue of spin, helicity, naturally arises in the form of the Dirac equation. In Jacobson and Schulman (1984), more work is done drawing connections with Brownian motion, and they discuss the dominant path contributions to the path integral in equation (19).

2.5. Generalization to the Full Dirac Equation

I want to briefly mention the generalization of Feynman's Checkerboard to 3+1 dimensions. Using multiple, more drawn out techniques, Jacobson (1984) derives the continuum 3+1 equivalent of the propagator in equation (36) for a particle that steps in discrete time steps from one position to anywhere on a sphere of radius $a = v\epsilon$ centered around that position. That is, Jacobson (1984) considers an arbitrary, yet constant, velocity v for each step of the particle. Jacobson (1984) obtains for this propagator:

$$K(\vec{x}_b, t_b; \vec{x}_a, t_a) = \frac{1}{8\pi^3} \int d^3k \, e^{i\vec{k}\cdot(\vec{x}_b - \vec{x}_a)} \, e^{-i(t_b - t_a)(\vec{\alpha}\cdot\vec{k} + \beta m)} \tag{40}$$

where $\vec{\alpha}$ is given by equation (3) and β is given by

$$\beta = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \tag{41}$$

where I is the 2x2 identity matrix.

Equation (40) is the Fourier transform of the exact retarded Dirac equation propagator in 3+1 dimensions for four-component spinors in the Weyl representation. Thus, equation (40) provides a physical process from which spin in the Dirac equation naturally arises. I will not repeat the lengthy derivation here. However, I make note of the following caveat, similar to the one I made in Section 2.2; in the derivation of equation (40), Jacobson (1984) arrives at the requirement that

$$v = a/\epsilon \geq 3^{1/2}c \tag{42}$$

in order for the sum over the paths to converge to the result. Thus, the derivation requires superluminal motion of the *massive* particle in each of the steps, as opposed to the luminal motion in the 1+1 Feynman Checkerboard. Again, however, I note that the propagator is only evaluated for $|\vec{x}_b - \vec{x}_a| < (t_b - t_a)$. Finally, I note that the integral in (40) is not evaluated explicitly, and no connections are drawn to the three-dimensional Ising model, which may be of interest for future work.

3. Complex Time Interpretation

In the last section of this paper, I turn to some new speculative thoughts on spin in relation to the 1+1 Feynman Checkerboard propagator. Motivated by the lack of an adequate picture of spin, I independently arrived at the conceptual notion that spin can be thought of as some yet-to-be-determined angular motion through time. Just as rest mass can be thought of as the momentum of a particle moving at the speed of light through time, I posited that spin was a related intrinsic particle attribute. I considered 'motion' of a particle through the complex time plane, with a quantized two-valued slope to represent spin-1/2. At this time, Sudip Chakravarty referred me to Feynman's Checkerboard as a similar and possibly related concept. Note, this conceptual reasoning is completely unjustified and future work is proposed below to further investigate its validity.

At this point, I note in Figure 1 that time-reversal symmetry does not necessarily hold (depending on the intepretation). That is, the particle does reverse its direction in time (becoming the equivalent antiparticle), but the helicity does not switch sign - a particle moving to the right remains moving to the right and vice-versa. However, if you interpret the helicity to be defined by the value of the slope of the space-time trajectory, then time-reversal symmetry is recovered.

Consider instead Figure 2, representing the path of a particle at rest in space, but moving through the complex time plane from one point on the real time axis to another with 'time momentum' mc. The slopes of the line segments are chosen without justification to be $\pm 45°$ in the complex time plane, as this will lead to an equivalence with the Feynman Checkerboard in a particular case and allows one to use the same tools for calculating the propagator. The direction a particle propagates along the imaginary time axis I associate with the sign of the spin.

I can now point out that the caveats mentioned earlier in the paper involving massive particle motion at the speed of light no longer apply, and there is no apparent violation of special relativity for motion of a particle at rest in space and moving through the complex time plane. Furthermore, time-reversal symmetry - $t \to -t$, corresponding to a rotation of π in the complex time plane - is maintained with how I have defined the spin. If we instead interpret spin as the slope of the trajectory in the complex time plane, the time-reversal symmetry is lost. The proper choice of interpretation I leave as an open question.

The propagator for a particle through the complex time plane is then evaluated identically as in Section 2.3, with the formal substitution of $t' = Re[t]$ and $x' = iIm[t]$. To keep the variables of the different models distinguishable, I have primed the variables from the original Feynman Checkerboard Propagator. Note that I also define $t = Re[t] + iIm[t]$. Then, with the formal substitutions, τ simplifies nicely to $|t|$, and we have the analogue of equation (24):

$$K(Re[t], Im[t]; 0, 0) = \frac{im}{2} \begin{pmatrix} \frac{it}{|t|} J_1(m|t|) & J_0(m|t|) \\ J_0(m|t|) & \frac{it^*}{|t|} J_1(m|t|) \end{pmatrix} \tag{43}$$

Note, that I have chosen $\tau = +|t|$ as opposed to $-|t|$ on the basis that $Re[t]$ is chosen to be greater than zero in the path integral. I have not yet fully considered the implications of the $\tau = -|t|$ solution.

Next, considering only paths that end on the real time axis (ie, $Im[t] = 0$), equation (43) reduces to equation (24) with $x = 0$. That is, the two different checkerboards yield identical propagators for a particle at rest, my primary result.

While this result may appear trivial since the particle considered is at rest, it does overcome the difficulty of having a massive particle moving at the speed of light as in Feynman's Checkerboard. Furthermore, it offers a different possible interpretation for spin, albeit unjustified. To further investigate the validity of this interpretation, I propose that future work be done in considering an imaginary time checkboard propagator for a relativistic particle not at rest (ie $\vec{p} \neq 0$) and not reversing direction. Due to the time constraints on writing this paper, I have not yet had time to fully take this into consideration.

With this proposed work, I believe it will be a useful exercise to determine if we again recover the propagator form of equation (36) for the 1+1 dimensional case and equation (40) for the 3+1 dimensional case, and if the evaluated propagator analogous to equation (24) produces independent spinor solutions to the Dirac equation. The proposed calculation is necessary to ascertain the validity of these inquiries, as we lose information about the full propagator by considering only particles at rest. Further work might be done then in looking at how the imaginary time checkerboard propagator relates to the tools of analytic continuation, particles other than spin-1/2, and classical limits as $\hbar \to 0$.

4. Conclusion

Spin remains a fundamental quantized observable in quantum mechanics lacking a definite physical explanation of origin, "like the grin of a Chesire cat" as stated by Ohanian (1986). Several possibilities have been detailed in this paper, both simple and complicated, but none completely rigorous and/or justified. Physicists readily accept spin as a quantity not amenable to physical explanation, as the mathematics and observable consequences are well-understood. However, although there are many other fundamental questions to be answered today in physics, I believe that an explanation for the origin of spin should not be a forgotten question. It may have conceptual and physical implications for future work in other areas such as grand unification and internal structure in composite objects such as the proton and neutron.

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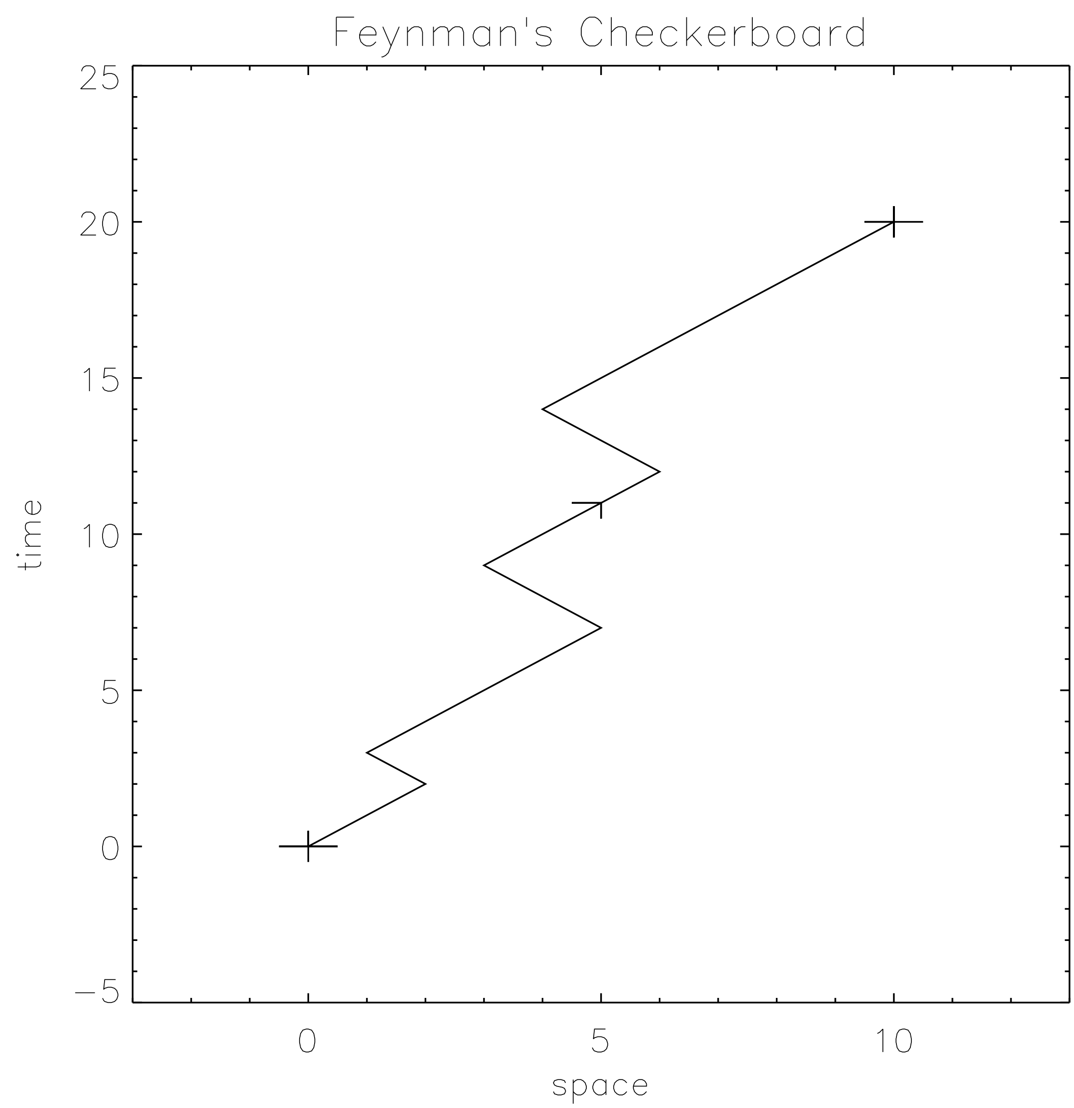

Fig. 1.— Feynman's Checkerboard. The particle moves at the speed of light in discrete steps from $(x, t) = (0, 0)$ to $(10, 20)$ in units of $\hbar = c = 1$ with $R = 6$ bends and $N = 20$ steps. The line segments are at $\pm 45°$, corresponding to a particle moving alternatingly left and right along the space dimension x at the speed of light. Since the particle starts moving to the right and ends moving to the right, this path contributes to K_{++}. This figure was generated using routines I wrote in IDL.

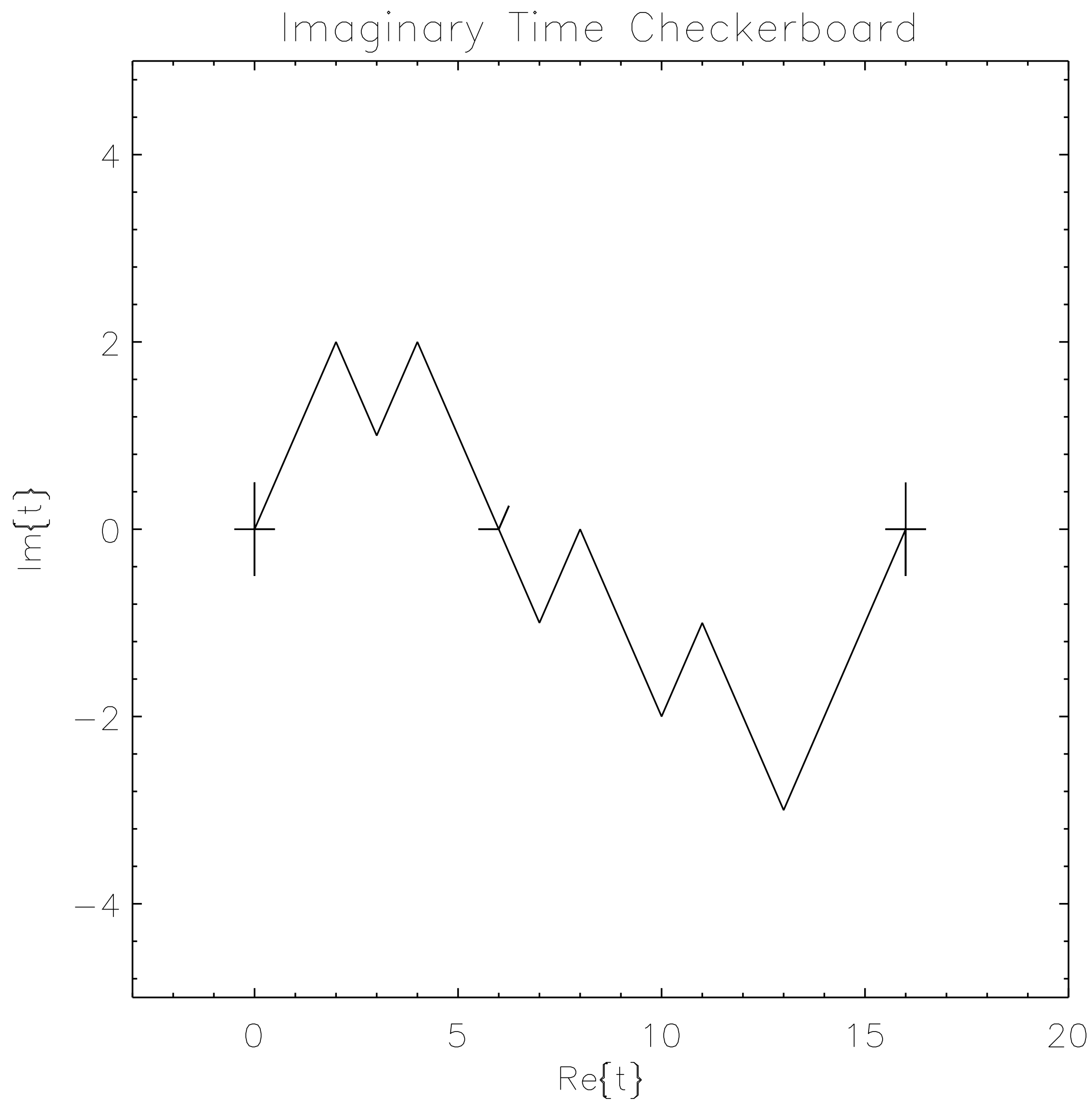

Fig. 2.— Complex Time Checkerboard. The particle moves through the complex time plane in $N = 16$ discrete steps from $(Re[t], Im[t]) = (0,0)$ to $(16,0)$,making $R = 8$ bends. The line segments are at $\pm 45°$. Since the particle starts moving in the positive imaginary time direction and ends the same, this path contributes to K_{++}. This figure was generated using routines I wrote in IDL.