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Qutrit Bound States, Mesons, and Baryons

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Abstract In the 1950s, Julian Schwinger defined what he called the “measurement algebra.” This was designed around the algebra of Stern-Gerlach experiments, that is, it described how Stern-Gerlach experiments connect together. We adapt his algebra to color bound states which we will call the “bound state algebra.”

The bound state algebra provides a natural method for computing approximate bound state energies. We provide two applications. The first application is the bound states and masses of the mesons. The second is to the generation structure of the elementary fermions and the Koide mass formulas for the leptons.

Keywords quantum theory · meson · baryon · generations · qubit · lepton

1 Introduction

The first major success of quantum field theory (QFT) was in calculating the Lamb shift; an adjustment to the energies of the hydrogen atom. In that calculation, QFT provides an adjustment to the energies calculated by quantum mechanics. Thus the Lamb shift calculation was (and still is) split between the older method of quantum mechanics and the newer method of quantum field theory.

Quantum mechanics applies to situations where the number of particles is constant in time. For the hydrogen atom, the particles involved are the proton and electron. QFT allows for the creation and annihilation of particles and so is much more general than quantum mechanics. Quantum mechanics glosses over the details of QFT by representing the force between particles with a

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potential energy function. For the Lamb shift, a $1/r$ potential energy is used just as with a classical electric field.

In the 1970s, the mesons were discovered to be composed of a quark and antiquark pair. A natural approach to calculating the meson masses is to follow the method of the Lamb shift calculation: first construct a bound state using quantum mechanics, and then compute adjustments using QFT. The particles that are constant would be the valence particles, the quark and antiquark.

The first problem with modeling the quarks with quantum mechanics is that we do not know what an appropriate potential energy would look like. It is believed that the quarks are asymptotically free, that is, at small distances, they act like unbound particles. But when they are widely separated, their potential energy is linear with distance making it impossible to separate them without creating new particles. Perhaps a suitable potential energy would be a infinite potential well. Rather than an infinite potential well, we will make an even simpler assumption; we will model the quarks with qubits.

Qubits are used in quantum information theory to represent the state of a single quantum bit. In their usual use they carry only spin information. For our purposes we will let them carry color instead of spin, or in addition to spin. We can do this with the mesons because the quark and antiquark are distinguishable particles and each can only be in one color state at a time (or superpositions). This is similar to the point particles of quantum mechanics in that a point particle can only be at a single point at a time (or superpositions). Therefore, the heart of our calculation will be in treating color as an abstract position variable.

One would suppose that it would be impossible to obtain any information about the energy of a bound state from a model that does not include nonzero momenta for the particles composing the bound state. However this is not entirely true. First, while qubits cannot carry any 3-momentum information, they can carry energy information and the condensed matter theorists using them in precisely that form.[2] And it is going to far to claim that we will derive their masses. Instead, as with the Regge trajectories, we will only derive a formula for the masses that must be fit to them.

In fitting the formula we will find that it works much better if one uses the square roots of the masses instead of their masses. This suggests that mass arises from the energy in a field and that it should be modeled as a vector.[1] We will apply this formula to meson resonances, baryon resonances, and the leptons and show that it works well.

The formula we derive turns out to be related to a formula that, in 1981, Yoshio Koide proposed for the charged lepton masses, a formula that precisely predicted the tau mass years before it was accurately measured. We will propose an explanation for this formula that extends it to the neutral leptons, that is the neutrinos. We will apply the same principle to quarks and derive a new principle for classifying the meson and baryon resonances.

These formulas will arise from modifying Schwinger's measurement algebra to apply to quantum bound states. Rather than solving linear equations as is usually done in quantum mechanics, we will solve bilinear equations, that is, systems of quadratic equations. Looking at the elementary fermions,

we will demonstrate a simple set of quadratic equations whose solutions are the weak hypercharge and weak isospin quantum numbers. From this model we will show how it comes to be that two of the quarks, the up and down, have masses that are approximately equal. In this model, it is not so much that the masses are equal, but the more difficult coincidence that their vectors are approximately equal.

2 Schwinger's Measurement Algebra

In the 1950s, Julian Schwinger developed an elegant but little noticed formulation of quantum mechanics he called the "measurement algebra." In this section we will introduce Schwinger's measurement algebra similar to the original paper[4], but with a more modern notation. As an example, we will consider the case of charged Dirac particles, the electron and positron. In addition, we will modify the notation to allow attributes that give three possibilities. Later we will need these to model color.

For microscopic systems, experiment has the difficulty that when one measures the system, the act of measurement changes the system. The effect of the measurement on the system is statistical in nature and it cannot be predicted in advance, except statistically. What's worse, a measurement of one attribute of the system can cause a change to another, previously measured, attribute of the system. An appropriate mathematical language for measurement needs to take this into account.

A good example of a quantum measurement is the Stern-Gerlach experiment. In this experiment, atoms are heated in an oven with a small hole. Atoms exit the small hole at random directions. A small, parallel sample of the atoms are passed through a magnetic field whose strength changes in the z direction. As a result of this, the atoms will separate into several groups according to their spin. For the present case, we will assume that the atoms have spin-1/2 and so there will be two spots:

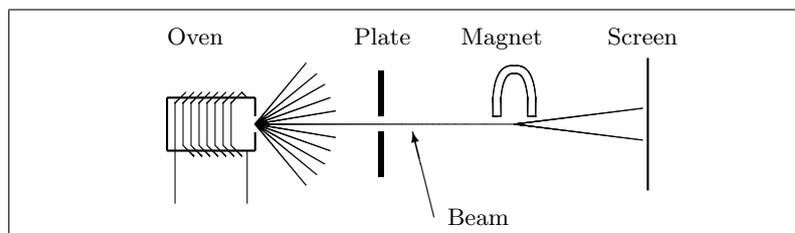


Fig. 1 The Stern-Gerlach experiment: Atoms are heated to a gas in an oven. Escaping atoms are formed into a beam by a small hole in a plate. A magnet influences the beam, which then forms a figure on a screen.

We wish to analyze what happens when the output of one Stern-Gerlach experiment (say, the top beam spot on the far right of Fig. (1)), is used as the input to another magnet assembly. For these purposes, we can assume

that the other beam, the bottom beam in the figure, is sent to a beam stop and does not participate further in the experiment.

For our purposes, we will define “measurement” as an operation on an ensemble of particles that keeps a portion of them according to some property. For the Stern-Gerlach experiment, the ensemble of particles is the incoming beam to the experiment. In the experiment as described above, the very first incoming beam comes from the oven. Later experiments can take as their incoming beam the beam output from a previous experiment.

In addition to measuring spin, we can measure other attributes of a particle beam. For a beam composed of a mixture of electrons and positrons, we could measure charge instead of spin. To measure charge, we will use some configuration of electric or magnetic fields that is different from the ones we used to measure spin but for our algebra the principle will be the same.

For measuring spin, let \mathbf{u} be a vector in 3-dimensions and suppose that a Stern-Gerlach experiment measures spin in this direction. We will label such a measurement U . Such a measurement allows only particles with spin measured to be $+1/2$ in the \mathbf{u} direction and eliminates all other particles. Spin-1/2 is a binary attribute in that the unpolarized beam is split into two. We can also contemplate ternary measurements, where the beam is split into three beams.

For these binary measurements, define the complementary measurement, the one that allows only particles with spin in the $-\mathbf{u}$ direction as \bar{U} . For the measurement that allows only positively charged particles to pass, we will use Q , and its complementary measurement, that allows only negatively charged particles, we will write \bar{Q} .

Measurements of spin and charge commute. That is, we can place these measurements in either order on the beam and the result will be the same. To relate our algebra to the ordering on the beam, we will place the earlier measurement on the right. So the compound measurement which consists of first taking the positively charged particles (with M_+) and then taking the spin up particles (with Z), will be written as $Z M_+$.

These measurements are “simple” in that they can be immediately repeated without altering the beam. That is, the beam that exists the measurement Q (or U) can be sent through another Q (or U) measurement and the output beam from that second measurement will be the same as the beam output from the first measurement. Algebraically, we say that Q (or U) is “idempotent.” This also applies to their complementary measurements:

$$\begin{aligned} Q Q &= Q, \\ \bar{Q} \bar{Q} &= \bar{Q}, \\ U U &= U, \\ \bar{U} \bar{U} &= \bar{U}. \end{aligned} \tag{1}$$

Another word for the above property is that these are “projection operators.”

Suppose an unpolarized beam of particles is sent through a series of measurements of the above type. The particles that survive the measurements will have certain properties in common. We are representing this experiment by a symbol. We can think of the symbols as being a representation (or description) of the particles that were picked out. In addition, we can also think

of the symbol as being a representation of the field configurations that were used in the experiment itself.

An “algebra” is a collection of symbols that include a 0 and a 1, and a definition of addition and multiplication. For the measurement algebra, we define 0 as a beam with no particles. We can create such a beam by multiplying two complementary measurements. For example, since no spin up particle is also spin down, and no positively charged particle is also negatively charged, we have:

$$\begin{aligned} U \bar{U} &= \bar{U} U = 0, \\ Q \bar{Q} &= \bar{Q} Q = 0. \end{aligned} \tag{2}$$

We define “1” as no measurement at all; the free beam. One can always multiply a measurement by 1 and we will leave the measurement unchanged.

We will assume that our symbols are associative, that is, given measurements A , B , and C , we have that $(AB)C = A(BC)$. For Stern-Gerlach experiments, it is difficult to say what a non associative algebra would imply. Perhaps it would mean that installing experiments A and B first, and then C would mean an experiment that acts differently than one where the second two experiments are installed first.

Addition is defined as the process that takes the outputs of two beams and combines them into a single beam. The complementary measurements add to 1:

$$\begin{aligned} U + \bar{U} &= \bar{U} + U = 1, \\ Q + \bar{Q} &= \bar{Q} + Q = 1. \end{aligned} \tag{3}$$

Like multiplication, addition is associative. And multiplication distributes over addition, so $A(B + C) = (AB) + (AC) = AB + AC$, and similarly for $(A + B)C$.

For a given quantum system, we will assume that there are only a finite number of attributes that are compatible, that is, that correspond to measurements that commute so that their order does not matter. For the example of the Dirac particles, we can take this to be spin in some direction, and charge. In this case, the maximum number of compatible attributes is 2. We can use this set of measurements, and their complements, to completely define a quantum state.

Suppose we have N compatible binary attributes. We can multiply together N measurements in a row for these measurements. Since each measurement is of a binary attribute, there are 2^N of these “primitive measurements” or “primitive idempotents.”¹ For the Dirac algebra, one can choose 4 primitive measurements. If one of the attributes were ternary instead of binary, then the number of primitive measurements would be a multiple of 3. For example, the up quark is a Dirac particle that has the additional ternary quantum number of color. Thus there are $3 \times 2^2 = 12$ primitive measurements for the up quark / anti-up quark system.

For the Dirac algebra, the traditional choice of attributes is spin and charge. For example, in an unpolarized beam of electrons and positrons,

¹ Schwinger calls these “elementary measurements,” we will follow the mathematical tradition.

there are spin up positrons, spin down positrons, spin up electrons and spin down electrons. These four measurements sum to unity:

$$\begin{aligned}
 & ZQ + \bar{Z}Q + Z\bar{Q} + \bar{Z}\bar{Q}, \\
 &= (Z + \bar{z})Q + (Z + \bar{Z})\bar{Q}, \\
 &= 1Q + 1\bar{Q}, \\
 &= Q + \bar{Q} = 1,
 \end{aligned} \tag{4}$$

and similarly for any other direction. In general, a “complete set of primitive measurements” or idempotents is a set of primitive measurements that use the same attributes and sum to unity. This is also true for ternary attributes.

Suppose that M is *not* a primitive measurement. Then there exists another attribute that can be used to split a beam defined by M . Call it A and assume that it is binary. Then $M A$ and $M \bar{A}$ are both nonzero measurements and, by distribution, we have $M = M A + M \bar{A}$. Therefore, any non primitive measurement can be written as a sum of nonzero measurements. Thus the mathematicians define primitive idempotents as idempotents that cannot be written as the sum of 2 nonzero idempotents. This definition works for ternary as well as binary attributes.

A complete set of primitive measurements has the convenient property that the product of any two different of them is zero. We say that they “annihilate.” This makes multiplication of the elements of such a set very simple. Given a set of such elements, $\{A_n\}$, the multiplication rule is

$$A_n A_m = \delta_n^m A_m, \tag{5}$$

where δ_n^m is the delta function, one where $n = m$ and zero otherwise.

In defining addition, we also have defined subtraction. Given $U + \bar{U} = 1$, we have $\bar{U} = 1 - U$. Unfortunately, subtraction does not have an immediate interpretation in the algebra. We can use subtraction to make calculations but there is no such thing as a negative beam. Or is there?

Suppose we have a primitive measurement P and some arbitrary, complicated (possibly including addition as well as multiplication) measurement M that does not commute with P . Consider a product that begins and ends with this primitive measurement $P M P$. Physically, such a product would consist of an experiment that begins and ends with the same primitive Stern-Gerlach measurement. As such, the effect on the beam will have to be proportional to the effect of the primitive measurement P . That is, the output of the measurement $P M P$ will be a beam of particles of type P , just as also characterizes the output of the measurement P . Furthermore, if the incoming beam is polarized, the first P of the PMP sequence will interact with the incoming beam in the same way as P and so PMP will, in a certain way, act just like P as far as multiplication goes. Just as $\bar{P} P = 0$, so $\bar{P} PMP = 0$. But we cannot say the same thing about addition as PMP could have a different effect on the amplitude and phase of the beam. Quantum mechanics uses complex numbers to describe wave amplitudes and phases so while PMP and P both describe similar particle types, they may be different in terms of amplitude and phase. We can write:

$$PMP = r e^{i\theta} P \tag{6}$$

where r and θ are real numbers. Complex numbers allow us to interpret a beam multiplied by a negative number, as well as the subtraction of beams.

Of particular interest in binary attributes is the difference between a measurement and its complement. We will write this as O_χ , for example, $O_U = U - \bar{U}$. As an element of the algebra, O_χ squares to unity. For example:

$$\begin{aligned} O_U^2 &= (U - \bar{U})^2, \\ &= UU - U\bar{U} - \bar{U}U + \bar{U}\bar{U}, \\ &= U - 0 - 0 + \bar{U}, \\ &= 1. \end{aligned} \tag{7}$$

In the usual formalism of quantum mechanics, O_χ is called the operator. If we have two attributes that are independent, for example spin U and charge Q , then their operators also commute $O_U O_Q = O_Q O_U$ as the operators are built from the measurements. We can also reverse the relationship and define the measurements from the operators:

$$\begin{aligned} U &= (1 + O_U)/2, \\ \bar{U} &= (1 - O_U)/2. \end{aligned} \tag{8}$$

For ternary attributes, similar relations obtain. For example, if R , G , and B are three complementary ternary measurements, then $R + G + B = 1$, $RG = RB = GB = 0$, $RR = R$, $GG = G$, $BB = B$, and O_R is defined by $O_R = R - G - B$. As in the binary case, $O_R^2 = 1$.

The number of particles (and therefore the energy) in a beam (or wave) is proportional to the squared magnitude of the amplitude. Taking the magnitude removes the arbitrary phase; the reason for squaring the magnitude is more difficult to explain. One could possibly rewrite all of quantum mechanics so that amplitudes were done away with but they are very convenient for computing interference problems as interference is linearly proportional to the amplitude. We use amplitudes because amplitudes allow linear superposition; the more natural quantities, energies, particle numbers, and beam densities do not.

Schwinger's measurement algebra is inherently nonlinear, that is, it is bilinear. Given two primitive measurements, their sum is generally not a primitive measurement. And the measurement algebra does not carry arbitrary complex phases. In Schwinger's original papers, and in the textbook introduction to quantum mechanics written from his notes,[?] the measurement algebra is brought into a form where it is linear. Schwinger does this by postulating a fictitious state which he calls the vacuum.[5] To obtain a creation operator, he assumes a measurement that takes the vacuum as input and produces a primitive measurement state as its output. Similarly, the annihilator takes a primitive measurement and changes it to the vacuum. In this paper we will avoid the quantum vacuum, and by doing this, work in a version of the measurement algebra where there are no arbitrary complex phases.

Let $\mathbf{u}_\theta = (0, \sin(\theta), \cos(\theta))$ and define QU_θ as the related primitive measurement. When $\theta = 0$ we have $U_0 = U_Z$, and when $\theta = \pi/2$ we have $U_{\pi/2} = U_Y$. Consider the product $QU_0 QU_\theta QU_0$. This corresponds to three consecutive Stern-Gerlach experiments with the center experiment offset by

the angle θ . This experiment has been performed. The effect of the experiment on the beam is to decrease its amplitude by multiplying it by a factor $(1 + \cos(\theta))/2$. This factor is 1 when $\theta = 0$ and 0 when $\theta = \pi$ so that the beam is unaffected by a repeat of the same Stern-Gerlach experiment (when $\theta = 0$) but is completely extinguished (which we call annihilated) by consecutive complementary measurements (when $\theta = \pi$).

Since U_θ and $U_{\theta+\pi}$ are complementary measurements, they sum to 1. We can write:

$$P = PP = P(U_\theta + U_{\theta+\pi})P = PU_\theta P + PU_{\theta+\pi}P. \quad (9)$$

The amplitude for $PU_{\theta+\pi}$ is $(1 + \cos(\theta + \pi))/2 = (1 - \cos(\theta))/2$. This is what we expect because $(1 + \cos(\theta))/2$ and $(1 - \cos(\theta))/2$ sum to 1.

The similarity of the experimental results for the amplitudes, $(1 \pm \cos(\theta))/2$, and the $(1 \pm U_\chi)/2$ when one defines the measurements in terms of the operators as shown in Eq. (8) suggests a method of defining a measurement in an arbitrary direction from the X , Y , and Z measurements. Given a unit vector $\mathbf{u} = (u_x, u_y, u_z)$ we compute U as follows:

$$\begin{aligned} O_X &= X - \bar{X}, \\ O_Y &= Y - \bar{Y}, \\ O_Z &= Z - \bar{Z}, \text{ and} \\ O_U &= u_x O_X + u_y O_Y + u_z O_Z, \\ U &= (1 + O_U)/2. \end{aligned} \quad (10)$$

The O_X , O_Y , O_Z are equivalent to the sigma matrices of the Pauli algebra with i defined as $i = O_X O_Y O_Z$. Showing that O_U squares to unity will depend on $u_x^2 + u_y^2 + u_z^2 = 1$ and the anticommutation relations of O_X , O_Y , and O_Z . It's not obvious how to show anticommutation, at least to this author.

The rules we've given for the measurement algebra, associativity, distribution of multiplication over addition, 0 and 1, commutativity for addition but not multiplication, are the same rules that one finds for matrix arithmetic. It is natural to create matrix representations of a measurement algebra.

For the spin operators, the usual choice of matrix representation is the Pauli algebra. The operators are defined as:

$$\begin{aligned} O_X & & O_Y & & O_Z \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (11)$$

The corresponding measurements, X , \bar{X} , Y , \bar{Y} , Z , and \bar{Z} are given by $(1 \pm O_U)/2$. The reader may recognize these measurement matrices as the density operators for the pure quantum states. For the Dirac algebra, one requires 4×4 matrices.

3 Hermiticity

Suppose we have two incompatible primitive measurements. What is the character of their product? Let P and P' be two primitive measurements. Consider the square of their product:

$$(P P')^2 = (P P' P) P'. \quad (12)$$

As noted above, a product that begins and ends with the same primitive measurement is a complex multiple of that primitive measurement. Therefore the term in parentheses, $P P' P$, is a complex multiple of P . Let $P P' P = kP$. We have:

$$\begin{aligned} (P P')^2 &= (P P' P) P', \\ &= k P P' \end{aligned} \quad (13)$$

Thus $P P'$ is almost idempotent. In fact, if we divide $P P'$ by k we obtain an idempotent:

$$(P P'/k)^2 = (P P'/k). \quad (14)$$

Furthermore, $P P'/k$ is primitive. Thus, the product of two primitive measurements is either zero, or a complex multiple of a primitive measurement. Of course the same cannot be said of a sum or difference of two primitive measurements.

As an example of the product of two different measurements that do not annihilate, consider the product of the measurements for spin in the $+z$ direction with the measurement for spin in the $+x$ direction. With the usual Pauli matrices, the product is as follows:

$$\begin{aligned} &Z X, \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (15)$$

In the above, $k = 1/2$. It is easy to see that the matrix to the right of the $1/2$ is, indeed, idempotent and primitive (as its trace is 1). However, unlike the usual density matrices, this matrix is non Hermitian (that is, the diagonal elements are not complex conjugates of each other). Other products, such as XY and YZ will also define non Hermitian measurement matrices.

This example makes the physical interpretation of a non Hermitian measurement clear. Such a measurement is obtained by two (or more) consecutive measurements with axes in non parallel directions. This is a perfectly reasonable compound measurement; it is not one that we can reject as non physical. The lack of Hermiticity is equivalent to noting that the measurement is not symmetric with respect to time; the order that a particle encounters the X and Z measurement depends on how one chooses the flow of time. When time is run backwards, the order of the measurements is reversed.

Given a complete set of Hermitian primitive measurements $\{M_n\}_{n=1}^N$, one can generally find another measurement (or more generally, any operator) X that is Hermitian, and does not annihilate any of the M_n . An example of $\{M_n\}$ is the measurements for spin-up and spin-down, and an example of a X

that does not annihilate any of these is spin-x. Given such a set of M_n and a non annihilating additional measurement X , we can define a convenient matrix representation of the entire algebra as we now show.

First, by assumption, all products of the form $M_j X M_k$ are nonzero. As noted above, these products are complex multiples of generally non Hermitian primitive measurements. There are N^2 of them, write them as

$$M_j X M_k = x_{jk} N_{jk} \quad (16)$$

where x_{jk} is a complex number and N_{jk} is a non Hermitian primitive measurement. These N^2 complex numbers, x_{jk} are the representation of X under our matrix representation. In general, given X and Y as arbitrary operators, this transformation into x_{jk} and y_{jk} preserves multiplication:

$$\begin{aligned} XY &= \sum_n X M_n Y = \sum_n X M_n M_n Y = \sum_n (X M_n)(M_n Y), \text{ and so} \\ (xy)_{jk} &= \sum_n M_j ((X M_n)(M_n Y)) M_k, \\ &= \sum_n x_{jn} y_{nk}. \end{aligned} \quad (17)$$

The transformation obviously preserves addition, unity, and zero, and so this is a representation of the algebra. In this representation, the measurements M_n are diagonalized; m_n is the matrix with 1 as the n th element on the diagonal and all other elements zero.

Likewise, given a matrix representation of a measurement algebra, we can find the measurements that define the representation; M_n is the primitive idempotent (and therefore measurement) with 1 in the n th position on the diagonal and all other elements zero. And X can be given by the ‘‘democratic’’ primitive idempotent, that is, the primitive idempotent matrix that has all elements equal. For an $N \times N$ matrix, the democratic primitive idempotent has all elements equal to $1/N$.

When one defines quantum mechanics using a Hilbert space, the operators are defined as Hermitian or non Hermitian according as they treat the inner product of the Hilbert space. A Hermitian operator M has matrix elements that satisfy:

$$\langle J|M|K\rangle = (\langle J|M|K\rangle)^*, \quad (18)$$

for $|J\rangle$ and $|K\rangle$ any state in a basis for the Hilbert space. As a definition of Hermiticity for a measurement algebra (or density matrix) formalism, the above is lacking in that the two sides of the equality are changed under phase changes of the bras and kets. Transforming $|J\rangle$ and $|K\rangle$ by $|J\rangle \rightarrow e^{ir}|J\rangle$ and $|K\rangle \rightarrow e^{ig}|K\rangle$ we end up with the above equation transforming as:

$$e^{i(g-r)} \langle J|M|K\rangle = (e^{i(r-g)} \langle K|M|J\rangle)^* = e^{i(g-r)} (\langle K|M|J\rangle)^*. \quad (19)$$

The complex phase, $\exp(i(g-r))$ on each side cancels so the definition is adequate for a Hilbert space. The measurement algebra does not have an inner product so the above cannot be defined that way. In addition quantum states in the measurement algebra do not have arbitrary phase. As a consequence, there is no simple construct in the measurement algebra that will define Hermiticity in an absolute sense. Our approach will be to assume a matrix representation of the algebra and define Hermiticity using the matrix definition.

Given a Hermitian operator X is defined as above, we have that its matrix elements are complex conjugates: $x_{jk} = x_{kj}^*$. Physically, this is a relationship between two different sequences of operations on the beam, $H_j X H_k$, and $H_k X H_j$. These sequences are different in that one is the time reversal of the other; the beam particles see the measurements in reverse order. In quantum mechanics, one represents the amplitude as a function of time with a factor of $\exp(+i\omega t)$. The action of the measurement sequence $H_j X H_k$ is to multiply this by x_{jk} to give $x_{jk} \exp(+i\omega t)$. On time reversing the wave function, the $+i\omega t$ becomes $-i\omega t$ and the x_{jk} becomes x_{kj} as the operations on the beam are reversed in order. Thus, in the measurement algebra, Hermiticity means an operator that is unchanged by time reversal. In addition, Hermitian primitive idempotents are primitive idempotents that can be created from vectors (or spinors).

Let J and K be two, Hermitian, primitive measurements that do not annihilate each other. Their product, JK must be a complex multiple of a primitive measurement. This procedure can be reversed; given any primitive measurement X , (which is Hermitian or not), it can be written in exactly one way as a complex multiple of the product of two Hermitian primitive measurements. To see this, first note that any idempotent operator has eigenvalues of 0 and 1 only, as the eigenvalues must also satisfy the complex idempotency equation $\lambda^2 = \lambda$. Primitive measurements have trace 1, and so the eigenvalue 1 has multiplicity 1 while the eigenvalue 0 has multiplicity $N - 1$.

The matrix X is not necessarily Hermitian so its left eigenvectors are not necessarily the same as its right eigenvectors. Take these two eigenvectors (which are spinor/vector quantum states in the usual quantum mechanics) and convert them into matrices by the usual technique. These matrices are Hermitian by construction. Their product has all the correct left and right eigenvectors of the matrix X and so is a complex multiple of X . It remains to show that the construction is not zero and is unique which we leave as an exercise for the reader.

4 Mutually Unbiased Bases

Quantum mechanics is usually defined in terms of a Hilbert space so that the quantum states are the vectors of the Hilbert space. The measurement algebra is slightly more primitive, in terms of a Hilbert space the measurements are operators on the Hilbert space but are not in the Hilbert space itself. Nevertheless, we can use concepts and examples from Hilbert space. In particular, a basis for a Hilbert space can be put into density matrix form; the result is a complete set of measurements for the corresponding measurement algebra. As an example, a basis for the Hilbert space with 2 dimensions is, in bra form, $\{(1, 0), (0, 1)\}$. These define a complete set of measurements for the Pauli algebra, i.e. spin up and spin down.

An operator is “unbiased” with respect to a basis if its matrix elements are all equal in magnitude. With the Pauli algebra, spin measurements in the $\pm x$ or $\pm y$ direction are unbiased with respect to the usual basis of spin up and spin down: $\{(1, 0), (0, 1)\}$; the transition probabilities are 1/2. Spin in the $+x$ and $-x$ directions also define a basis for the Pauli algebra, that is, the

bras $\{(1, 1)/\sqrt{2}, (-1, 1)/\sqrt{2}\}$ define measurements (that are operators) that are unbiased with respect to spin in the $\pm z$ direction. In fact, all the basis elements in spin $\pm x$ are unbiased with respect to spin $\pm z$ and vice versa. When a pair of bases have this property, we call them “mutually unbiased bases” or MUB.

One can imagine three or more basis sets that are mutually unbiased all around. It is known that given a Hilbert space of dimension N , the largest number of mutually unbiased bases that one can find is $N + 1$. Such a set of $N + 1$ mutually unbiased bases is called “complete.” For the Pauli algebra, the maximum number is $2 + 1 = 3$; this is achieved when one chooses three perpendicular measurements. For example, spin $\pm x$, spin $\pm y$, and spin $\pm z$ define a complete set of mutually unbiased bases for the Pauli algebra.

In a certain sense, a complete set of MUBs completely define the degrees of freedom that can be present in a quantum state. We need this fact and its demonstration is pedagogically useful, so suppose we have a beam of spin-1/2 particles but with unknown polarization, possibly a statistical mixture. What must we do to determine the density matrix representation of the beam? Statistical mixture density matrices conventionally have trace 1. Define the density matrix as follows:

$$\frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix}, \quad (20)$$

where (u_x, u_y, u_z) defines the Bloch vector for the mixed density matrix. We have $u_x^2 + u_y^2 + u_z^2 \leq 1$, with equality for the case of a pure density matrix.

If we send the beam into a spin-up measurement, the beam amplitude will be defined by the trace of the product of the density matrices:

$$\text{tr} \frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1 + u_z)/2. \quad (21)$$

Similarly, spin-down gives $(1 - u_z)/2$. This gives the terms on the diagonal but we have no information about the off diagonal terms. So let's send the beam through a measurement of spin in the $+x$ direction:

$$\text{tr} \frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (1 + u_x)/2. \quad (22)$$

Similarly for spin in the $-x$ direction. It remains to determine the u_y component which will require a third measurement, that of spin in the $\pm y$ direction. In quantum tomography, one uses measurement to determine all the information in a quantum state. Complete sets of mutually unbiased bases are the way this is done most efficiently; each measurement gives unique information that is not obtained in any of the other measurements.

In any single basis, the primitive measurements of the Schwinger measurement algebra have a rather dull and uninteresting multiplication table. Like jealous lovers, they simply annihilate each other. This is unfortunate because the action of a Stern-Gerlach experiment on a particle beam can be thought of as a force acting on the beam. As such, the action of the primitive measurements on each other can be thought of as forces. Having two measurements annihilate means that the particles do not interact.

Primitive measurements taken from two MUBs are more interesting. For the Pauli algebra, the complete set of MUBs has three bases so we can take one measurement from each basis and compute their multiplication table. We will now do this for the Pauli algebra.

We will take the measurements of spin-1/2 in the $+x$, $+y$, and $+z$ directions, that is, X , Y , and Z . As noted in the previous section, their cross products, XY , XZ , YX , YZ , ZX , and ZY are not Hermitian and so in writing out a closed multiplication table we must include these six states for a total of nine. Any further multiplications will give complex multiples of these nine; any product of X , Y , and Z can be expressed as a complex multiple of one of these nine.

Writing out the multiplication table for the products of these MUB elements would require quite some effort without an understanding of Berry-Pancharatnam phase. We will discuss this in the following section, but for the moment, we will borrow the techniques of that section and write out the closed multiplication table for a set of three elements taken one from each of the three basis sets of the complete Pauli MUB, the work of perhaps 15 minutes:

	X	Y	Z	XY	YZ	ZX	YX	ZY	XZ
X	X	XY	XZ	XY	vXZ	$X/2$	$X/2$	v^*XY	XZ
Y	YX	Y	YZ	$Y/2$	YZ	vYX	YX	$Y/2$	v^*YZ
Z	ZX	ZY	Z	vZY	$Z/2$	ZX	v^*ZX	ZY	$Z/2$
XY	$X/2$	XY	vXZ	$XY/2$	vXZ	$vX/2$	$X/2$	$XY/2$	$XZ/2$
YZ	vYX	$Y/2$	YZ	$vY/2$	$YZ/2$	vYX	$YX/2$	$Y/2$	$YZ/2$
ZX	ZX	vZY	$Z/2$	vZY	$vZ/2$	$ZX/2$	$ZX/2$	$ZY/2$	$Z/2$
YX	YX	$Y/2$	v^*YZ	$Y/2$	$YZ/2$	$YX/2$	$YX/2$	$v^*Y/2$	v^*YZ
ZY	v^*ZX	ZY	$Z/2$	$ZY/2$	$Z/2$	vZX	v^*ZX	$ZY/2$	$v^*Z/2$
XZ	$X/2$	v^*XY	XZ	$XY/2$	$XZ/2$	$X/2$	$v^*X/2$	v^*XY	$XZ/2$

(23)

where $v = (1 + i)/2$.

The nine elements, X , Y , Z , XY , etc., can be naturally placed in a matrix form with the X , Y , and Z primitive measurements on the diagonal, and the others appropriately placed in the other six locations. This is a matrix of primitive measurements rather than the usual matrix of complex numbers. It corresponds to various sequences of primitive measurements and we will use the matrix form to do quick calculations in what would otherwise be a difficult non commutative algebra.

The above table tells how multiplication of this matrix differs from multiplication of complex matrices. The top left quarter of the table shows that the products of X , Y , and Z with each other end up with no adjustments. This will mean that in translating a matrix of MUBs into a complex matrix we will leave the diagonal unchanged. We will provide the complete transformation of matrices of these Pauli MUBs into complex matrices in Sec. (12), where we will provide more motivation for the reader to suffer through it.

5 Berry-Pancharatnam Phase

Of particular interest to us are measurements that begin and end with the same primitive measurement. Such a sequence is a complex multiple of a Hermitian primitive measurement. For example, we find that $ZXYZ$ is given by:

$$\begin{aligned} ZXYZ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & -0.i \\ +0.5i & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ &= \frac{1+i}{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ &= (0.5)^{3/2} e^{i\pi/4} Z. \end{aligned} \tag{24}$$

In the above, the magnitude of the complex number k , that is, the $0.5^{3/2}$ can be attributed to the effect of three 90 degree transitions between different measurements. Only half the particles survive each such transition, so the amplitude is decreased by $\sqrt{0.5}$ three times. The complex phase, $\pi/4$, is the Berry-Pancharatnam phase, or quantum phase. For a more general product, the magnitude of the complex number k is given by the product of $\sqrt{(1 + \cos(\theta))/2}$ factors while the complex phase is given by half the area of the (oriented) spherical polygon whose vertices are defined by the spin vectors of the primitive measurements included in the product as we now show.

Let \mathbf{u} , \mathbf{v} , \mathbf{w} be three real unit vectors, no two antiparallel. We wish to show that $UVWU$ is equal to $e^{is_{uvw}}U$ where s_{uvw} is half the area of the oriented spherical triangle defined by \mathbf{u} , \mathbf{v} , and \mathbf{w} . First, note that since this is a problem written in the measurement algebra (or density matrices) there are no arbitrary complex phases and so the problem is well defined, the phase is a physical quantity. For any given spherical triangle, there is a unique phase associated with it, possibly dependent on the order in which the points are listed. More generally, given any product of non antiparallel measurements that begins and ends with the same measurement, there is a unique phase associated with it.

Let $UVWU = kU$ and consider $VUVWUV = V(UVWU)V = kVUV$. Since VUV is a real multiple of V , we have that the phase of $UVWU$ is the same as the phase of $VUVWUV$. However, we can write $VUVWUV = (VUV)WUV$, and since VUV is a real multiple of V , we have that the phase of $VUVWUV$ is the same as $VWUV$ and the phases of $UVWU$ and $VWUV$ are equal. Thus we have shown that the phase is a property of the triangle that does not depend on which order the vertices are listed (provided that we consider only cyclic permutations). It is also easy to show that swapping two points on the triangle will negate the phase. Thus we have that phase is a property of oriented areas. We next show that it is additive.

Let $UVWU = kU$ so that $s = \text{Phase}(k)$. Now consider a 4th unit vector, \mathbf{x} , also not antiparallel to any of the others, and its measurement operator X . Let $(UVWXVU)(UVXU)(UXWU) = k'U$ and $s' = \text{Phase}(k')$. We now consider a sequence of algebraic operations that we can perform on the product $(UVWXVU)(UVXU)(UXWU)$ which do not change the phase s' . These operations consist of two sorts of things, first, we can replace products like UU with U because all the measurements are idempotent. Second, any

time we have one primitive measurement between two copies of the same primitive measurement, such as UVU , we can replace that triple with just the primitive measurement on the outside because their phases will be the same. Then the following complex multiples of U all have the same phase:

$$\begin{aligned}
& (UVWXVU)(UVXU)(UXWU), \\
& \equiv UVWX(VUV)(XUX)WU, \\
& \equiv UVW(XVX)WU, \\
& \equiv UV(WXW)U, \\
& \equiv UVWU.
\end{aligned} \tag{25}$$

And we have that $s = s'$. As shown before, the phase of $UVWXVU$ is the same as the phase of $VWXV$. But if k_1 is the phase of $UVWXVU$, and k_2 is the phase of $UVXU$, and k_3 is the phase of $UXWU$, then we have that the phase of $UVWU$ is equal to the phase of $k_1 k_2 k_3$ which is equal to the sum of the phases of k_1 , k_2 , and k_3 . But if X is inside the oriented triangle $UVWU$, then this triangle is composed of the three triangles $UVXU$, $UXWU$, and $VWXV$. Therefore, phase must be proportional to area. Finally, we choose an infinitesimal triangle, for example, one with vertices $(0, 0, 1)$, $(a, 0, 1)/\sqrt{1+a^2}$, and $(0, a, 1)/\sqrt{1+a^2}$, and compute its area and phase. From this we find that the constant of proportionality is $1/2$. The factor of $1/2$ is natural in that it scales the total surface area of the sphere, 4π , down to the 2π length of the unit circle.

While the measurement algebra is nonlinear in that it does not satisfy the linear superposition principle, it does have an addition defined. So it is more correct to say that its linearity is a different sort of linearity from that usually exhibited in quantum mechanics. Problems which are linear in the usual quantum mechanics become difficult nonlinear problems in the measurement algebra. And problems that are linear in the measurement algebra can be quite difficult in the usual state vector form of quantum mechanics. In particular, phase problems, such as the Berry-Pancharatnam just discussed are particularly simple in the measurement algebra. Berry-Pancharatnam phase was not discovered until the 1950s.

The bound state algebra, the density matrix formalism, and Schwinger's measurement algebra are similar in that each is a bilinear representation of a quantum state. Bilinear formalisms are similar in that they represent a quantum state by the product of a ket $|\psi\rangle$, and a bra $\langle\psi|$. Bras and kets are subject to the arbitrary complex phases of quantum mechanics in an opposite way:

$$\begin{aligned}
|\psi\rangle & \rightarrow e^{+i\delta}|\psi\rangle, \\
\langle\psi| & \rightarrow e^{-i\delta}\langle\psi|.
\end{aligned} \tag{26}$$

On multiplication, the opposite signs in the exponentials cancel. Thus bilinear representations are unchanged under such transformations.

Suppose we have three distinct quantum states, $|R\rangle$, $|G\rangle$, and $|B\rangle$. Let's consider a transformation of their complex phases that modifies each by a different angle:

$$\begin{aligned}
|R\rangle & \rightarrow e^{+ir}|R\rangle, \\
|G\rangle & \rightarrow e^{+ig}|G\rangle, \\
|B\rangle & \rightarrow e^{+ib}|B\rangle,
\end{aligned} \tag{27}$$

with the bras carrying the negative phases. Since the complex phases are arbitrary, no physical observable can be modified by the above transformation. Thus a matrix elements, such as $\langle R|M|G\rangle$, is not a physical observable (as it takes a complex phase of $g - r$ under the above transformation). The complex conjugate, $\langle G|M|R\rangle$, takes the negative phase, $-(g - r)$. Consequently the product of these two matrix elements, $\langle R|M|G\rangle\langle G|M|R\rangle = |\langle R|M|G\rangle|^2$ is unchanged by the transformation and is the observable associated with the matrix element. Given a general product of matrix elements, the effect of arbitrary complex phase changes will be to possibly change the phase of the product. The products of matrix elements that correspond to observables are those that do not depend on arbitrary complex phase. This definition of observable is slightly more general than the common but naive observation that observables must always be real.

A product of matrix elements will avoid having arbitrary complex phase if each quantum state included includes just as many bras as kets. The simplest non trivial example of such a product is:

$$\langle R|G\rangle\langle B|R\rangle\langle G|B\rangle \quad (28)$$

In the above, just as many factors of $|\psi\rangle$ are included as $\langle\psi|$ and so the above is unchanged by complex phase. However, unlike $\langle R|G\rangle\langle G|R\rangle$, the above is not necessarily real. In general, it will be complex. To see this, let $|R\rangle$, $|G\rangle$, and $|B\rangle$ be eigenvectors for spin in the $+x$, $+y$, and $+z$ directions:

$$\begin{aligned} \langle R| &= (1, 1)/\sqrt{2}, \\ \langle G| &= (1, i)/\sqrt{2}, \\ \langle B| &= (1, 0). \end{aligned} \quad (29)$$

And the product $\langle R|G\rangle\langle B|R\rangle\langle G|B\rangle$ is $1 + i = \sqrt{2}e^{i\pi/4}$, an example of Berry-Pancharatnam phase.

A product of matrix elements that includes just as many bras as kets can be rearranged so that each ket appears next to its corresponding bra. This can be accomplished since each of the matrix elements is just a complex number and commutes with everything else. The product $\langle R|G\rangle\langle B|R\rangle\langle G|B\rangle$ can be rearranged to $\langle R|G\rangle\langle G|B\rangle\langle B|R\rangle$. One can then turn the product into a product of traces of density matrices:

$$\begin{aligned} \langle R|G\rangle\langle B|R\rangle\langle G|B\rangle &= \langle R|G\rangle\langle G|B\rangle\langle B|R\rangle, \\ &= \text{tr}(\langle R|G\rangle\langle G|B\rangle\langle B|R\rangle), \\ &= \text{tr}(|R\rangle\langle R|G\rangle\langle G|B\rangle\langle B|), \\ &= \text{tr}(|R\rangle\langle R| |G\rangle\langle G| |B\rangle\langle B|), \\ &= \text{tr}(\rho_R \rho_G \rho_B). \end{aligned} \quad (30)$$

Consequently, a way of distinguishing observables from arbitrary functions of matrix elements is that observables can always be put into density matrix form.

6 Quantum Statistics

With multiparticle states, one can consider the operator S that swaps the quantum states of two particles; for example:

$$S|\alpha, \beta\rangle = |\beta, \alpha\rangle \quad (31)$$

This operator squares to unity and so can have eigenvalues of ± 1 only. When the exchanged quantum states are identical, it is a postulate of quantum mechanics that observed states must be eigenstates of S and so must have eigenvalues of $+1$ (bosons) or -1 (fermions). We say that bosons are symmetric under the exchange of any two states while fermions are antisymmetric.

When the exchange operator S is applied to a bilinear representation of a quantum state it results in a factor of ± 1 for the bra and another ± 1 for the ket. The sign depends on whether the particles are fermions or bosons but in either case it appears twice; the product will be $+1$ for both the fermion and the boson cases. Thus bilinear representations of quantum states with identical particles are always unchanged under exchange. Similarly the minus sign that spinors pick up on rotation by 360 degrees does not appear in bilinear representations and is not necessarily a part of the physical world.

Given a multiparticle state with more than just two identical particles, we can consider permutations more complicated than the exchange of two particles. For example, we can consider a cyclic permutation of a three particle state:

$$C|\alpha, \beta, \gamma\rangle = |\beta, \gamma, \alpha\rangle. \quad (32)$$

Since $C^3 = 1$, its eigenvalues can only be cubed roots of one, $e^{2in\pi/3}$. Suppose we have a multiparticle state that is an eigenstate of C . If the state is completely symmetric (or anti symmetric) under the exchange of two states, then, since we can write C as the product of two swaps, the only possible eigenvalue C can take will be $+1$ and the complex eigenvalues $e^{\pm 2i\pi/3}$ will be impossible. This is a consequence of the postulate that all quantum states are either symmetric or antisymmetric under the swap of identical particles. While this postulate works well for the observed particles, it is only a postulate and we need not follow it for a model of particles that are beyond the standard model; this paper deals with preon states that can carry complex eigenvalues under permutations.

7 The Bound State Algebra

In approaching the problem of representing bound states of qubits, we will look for an algebra whose rules are similar to the rules of the measurement algebra. That is, we would like the quantum states to be represented by objects that form an algebra that can be represented in matrices. Ideally, we would like the quantum states to be idempotent, and for incompatible states to annihilate each other.

For a quantum mechanical bound state problem, one typically defines a number of compatible operators and one finds a complete set of wave functions that have quantum numbers defined by these operators. For the case

of the quantum mechanical approximation of the hydrogen atom, a complete set of operators consists of the energy H , the angular momentum in the z direction, L_z , and the total squared angular momentum $L^2 = L_x^2 + L_y^2 + L_z^2$. We will scale these operators so that their eigenvalues are $-1/n^2$, m , and $l(l+1)$, respectively and label the states with them as $\psi_{nlm}(x, y, z, t)$. Note that the wave functions are solutions of a time independent Schroedinger's equation and do not depend explicitly on time.

The eigenvalue equations for the hydrogen bound states are:

$$\begin{aligned} H\psi_{nlm} &= -1/n^2 \psi_{nlm}, & n=1,2,\dots \\ L^2\psi_{nlm} &= l(l+1) \psi_{nlm}, & l=0,1,\dots,n-1 \\ L_z\psi_{nlm} &= m \psi_{nlm}, & m=-l,-l+1,\dots,l. \end{aligned} \quad (33)$$

The operators are assumed to be Hermitian. Taking the complex conjugate of the above equations, we have:

$$\begin{aligned} \psi_{nlm}^* H &= -1/n^2 \psi_{nlm}^*, & n=1,2,\dots \\ \psi_{nlm}^* L^2 &= l(l+1) \psi_{nlm}^*, & l=0,1,\dots,n-1 \\ \psi_{nlm}^* L_z &= m \psi_{nlm}^*, & m=-l,-l+1,\dots,l. \end{aligned} \quad (34)$$

We will not be concerned with the details of the above, except to note that any two different hydrogen wave functions, say ψ_{nlm} and $\psi_{n'l'm'}$, have at least one quantum number different, and that they are eigenfunctions on either side of the operators.

The wave functions are normalized so that the integral of their squared magnitude gives one:

$$\int \psi^*(x) \psi(x) d^3x = 1. \quad (35)$$

One computes the average of an operator (such as the H , L^2 , L_z , etc.) by using an integral over all space as follows:

$$\langle O \rangle(t) = \int \psi^*(x, t) O \psi(x, t) d^3x \quad (36)$$

where the integral is over all space. For the eigenstates and the H , L^2 , and L_z operators these reduce to $-1/n^2$, $l(l+1)$, and m , respectively.

In the above integrals, the wave function is inserted into the formula twice. We can also imagine inserting two possibly different wave functions, say ψ_{nlm} and $\psi_{n'l'm'}$ into the above formula. We will label such a construction as follows:

$$\langle nlm | O_\chi | n'l'm' \rangle = \int \psi_{nlm}^* O_\chi \psi_{n'l'm'} d^3x \quad (37)$$

If the two wave functions are different, they must differ in their eigenvalue for at least one of the three operators H , L^2 , or L_z . Without loss of generality, assume that they differ in their H eigenvalue. Then compute:

$$\begin{aligned} \langle nlm | H | n'l'm' \rangle &= \int \psi_{nlm}^* (H \psi_{n'l'm'}) d^3x, \\ &= (-1/n'^2) \int \psi_{nlm}^* H \psi_{n'l'm'} d^3x. \end{aligned} \quad (38)$$

On the other hand, per Eq. (34), we can also apply the operator to the left. Then the calculation gives:

$$\begin{aligned} \langle nlm|H|n'l'm'\rangle &= \int (\psi_{nlm}^* H) \psi_{n'l'm'} d^3x, \\ &= (-1/n^2) \int \psi_{nlm}^* H \psi_{n'l'm'} d^3x. \end{aligned} \quad (39)$$

Since these are equal, and since $n \neq n'$, these two calculations for the same integral will give different results unless both answers are zero. Therefore, remembering the normalization given in Eq. (35), we have:

$$\langle nlm|n'l'm'\rangle = \int \psi_{nlm}^* \psi_{n'l'm'} d^3x = \delta_n^{n'} \delta_l^{l'} \delta_m^{m'}. \quad (40)$$

That is, the eigenfunctions are orthonormal. Furthermore, this result is perfectly general for any set of bound states.

The above defines an inner product on the wave functions. This is unlike the multiplication of Schwinger's measurement algebra in that the result of multiplying two wave functions is just a complex number, not another wave function. To turn the inner product into a multiplication function on the quantum states we must switch to a form of the density matrix representation.

Let ψ_{nlm} be a wave function. We define the "bound state operator" B_{nlm} as follows:

$$B_{nlm}(x, t, x', t') = \psi_{nlm}(x, t) \psi_{nlm}^*(x', t'). \quad (41)$$

Note that in the above, the complex conjugate is on the right hand side. The product of two of these sorts of things, arranged so that the two inner wave functions use the same position and time coordinates, will put adjacent wave functions together so that the complex conjugate will be on the left, as used in the above equations. See the quantity in parentheses below:

$$\begin{aligned} B_{nlm}(x, t, x', t') B_{n'l'm'}(x', t', x'', t'') &= \psi_{nlm}(x, t) \psi_{nlm}^*(x', t') \psi_{n'l'm'}(x', t') \psi_{n'l'm'}^*(x'', t''), \\ &= \psi_{nlm}(x, t) (\psi_{nlm}^*(x', t') \psi_{n'l'm'}(x', t')) \psi_{n'l'm'}^*(x'', t''). \end{aligned} \quad (42)$$

So to define the product of two B_χ operators, we integrate over the intermediate position data:

$$B_{nlm} B_{n'l'm'}(x, t, x'', t'') = \int \psi_{nlm}(x, t) \psi_{nlm}^*(x', t') \psi_{n'l'm'}(x', t') \psi_{n'l'm'}^*(x'', t'') d^3x'. \quad (43)$$

Note that the above could depend on t' . We could avoid this by also integrating over t' , but for our purposes there is no reason to. The wave functions we will be using will be presumed to be time independent. Consequently, the right hand side does not depend on t' . We can now leave off the space and time coordinates (x, t, x', t') and treat products of these operators as a multiplication on the wave functions. It is straightforward to verify that this product is associative.

The addition of the bound state operators is the usual addition. Given two bound state operators, B , and C , we define the operator for their sum as:

$$(B + C)(x, t, x', t') = B(x, t, x', t') + C(x, t, x', t'). \quad (44)$$

Addition is clearly associative. The 0 of the algebra is the operator which is always zero, and the 1 is the operator which is always 1. It is also easy to see that multiplication distributes over addition. Therefore, we have defined an algebra of the bound states.

From the orthonormality of the wave functions, we see that the bound state operators are idempotent:

$$\begin{aligned}
 B_{nlm}B_{nlm}(x, t, x'', t'') &= \int \psi_{nlm}(x, t) \psi_{nlm}^*(x', t') \psi_{nlm}(x', t') \psi_{nlm}^*(x'', t'') d^3x', \\
 &= \psi_{nlm}(x, t) \left(\int \psi_{nlm}^*(x', t') \psi_{nlm}(x', t') d^3x' \right) \psi_{nlm}^*(x'', t''), \\
 &= \psi_{nlm}(x, t) \psi_{nlm}^*(x'', t''), \\
 &= B_{nlm}(x, t, x'', t'').
 \end{aligned} \tag{45}$$

Replacing the second B_{nlm} with $B_{n'l'm'}$ we find that bound states made from orthogonal wave functions annihilate each other.

It is also possible to write down bound states that are different but are not orthogonal. As with Schwinger's measurement algebra, the product of such a pair will be a complex multiple of a generally non Hermitian bound state operator. For the case of Schwinger's measurement algebra, the interpretation of non Hermitian operators was that they corresponded to measurements that were not symmetric in time. This was natural in that the measurement algebra was defined as a model for beams which encounter measurements in a time ordering. We defined the bound states with two different times, t and t' . Using different times makes these objects similar to scattering matrices, but with finite times. Therefore, we can interpret a non Hermitian bound state as arising from a bound state where the wave function goes through a sequence of states that is not time symmetric. Such a bound state would violate time reversal symmetry. Perhaps non Hermitian bound states may be of use in modeling a bound state whose decays violate time reversal symmetry or charge-parity (CP) such as the η or the B mesons.

Suppose that a wave function ψ happens to be restricted to just three points, say $x = 1$, $x = 2$, and $x = 3$. There will be three probabilities for the particle to be found at these three points, call them p_n . By normality, $p_1 + p_2 + p_3 = 1$. In order for our continuous wave function to give these results, we could have

$$\begin{aligned}
 \psi(x, t) &= \sqrt{p_1} \delta(x-1) + \sqrt{p_2} \delta(x-2) + \sqrt{p_3} \delta(x-3), \\
 &= \sum_{n=1}^3 \beta_n \sqrt{\delta(x-n)},
 \end{aligned} \tag{46}$$

where $\delta(x)$ is Dirac's delta function. The square roots are necessary since the wave function must be squared in order to give an amplitude. In general, the amplitudes β_n may be complex. On converting this wave function to bound state operator form, we have:

$$\begin{aligned}
 B &= \sum_{n=1}^3 \sum_{m=1}^3 \beta_n \beta_m^* \sqrt{\delta(x-n)\delta(x'-m)}, \\
 &= \sum_{n,m} b_{nm} \sqrt{\delta(x-n)\delta(x'-m)},
 \end{aligned} \tag{47}$$

where we have introduced $b_{nm} = \beta_n \beta_m^*$, a matrix that contains the information about this wave function.

Suppose we have two bound state operators B and C with amplitude matrices given by b_{nm} and c_{nm} . Then the product BC has an amplitude matrix given by the matrix product:

$$\begin{aligned}
BC(x, x'') &= \int \Sigma_{n,m} b_{nm} \sqrt{\delta(x-n)\delta(x'-m)} \Sigma_{n',m'} c_{n'm'} \sqrt{\delta(x'-n')\delta(x''-m')} d^3x', \\
&= \Sigma_{n,m,n',m'} b_{nm} c_{n'm'} \sqrt{\delta(x-n)} \int \sqrt{\delta(x'-m)\delta(x'-n')} d^3x' \sqrt{\delta(x''-m')}, \\
&= \Sigma_{n,m,n',m'} b_{nm} c_{n'm'} \sqrt{\delta(x-n)\delta_{n'}^m} \sqrt{\delta(x''-m')}, \\
&= \Sigma_{n,m'} \Sigma_{n'} b_{nn'} c_{n'm'} \sqrt{\delta(x-n)\delta(x''-m')}, \\
&= \Sigma_{n,m'} (bc)_{nm'} \sqrt{\delta(x-n)\delta(x''-m')}.
\end{aligned} \tag{48}$$

Similarly, addition for the discrete operators follows matrix addition, etc., and we can use standard matrix arithmetic to make calculations for bound state operators.

The simplest case is a scalar particle that is in a bound state on two discrete points so the matrices are 2×2 . The idempotents are easy to solve. Let a , b , c , and d be four complex numbers and compute:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{49}$$

This gives four coupled quadratic equations:

$$\begin{aligned}
a &= a^2 + bc, \\
b &= b(a + d), \\
c &= c(a + d), \\
d &= d^2 + bc.
\end{aligned} \tag{50}$$

If $a \neq d$, the middle two equations give $b = c = 0$ and there are two solutions, $a = d = 0$ and $a = d = 1$. These are the 0 and 1 matrices which are not primitive (because their traces are 0 and 2, respectively). The primitive matrices have trace 1 so $a + d = 1$ and the middle two equations are solved for any b and c . So we use b and c to parameterize the general solution. The solution is a little more elegant if we replace b and c by $b/2$ and $c/2$:

$$\frac{1}{2} \begin{pmatrix} 1 \pm \sqrt{1 - bc} & b \\ c & 1 \mp \sqrt{1 - bc} \end{pmatrix}. \tag{51}$$

If we require that the primitive idempotent be Hermitian, then bc must be real and between 0 and 1. In this case, our general solution amounts to a point on the Bloch sphere. That is, given a unit vector so that $u_x^2 + u_y^2 + u_z^2 = 1$, write $b = u_x - iu_y$ and $c = u_x + iu_y$; so $0 \leq bc \leq 1$. Then the pure density matrix for spin in the (u_x, u_y, u_z) direction is given by:

$$\frac{1}{2} \begin{pmatrix} 1 + u_z & u_x - iu_y \\ u_x + iu_y & 1 - u_z \end{pmatrix} \tag{52}$$

and $a = (1 + u_z)/2$, etc.

The symmetry of the standard model elementary particles is $U(1) \times SU(2) \times SU(3)$. In the standard model, two representations of $SU(2)$ are used, a spin-0 representation for the right handed particles, and a broken

spin-1/2 representation for the left handed particles. We've just seen that the Hermitian bound state algebra for a scalar particle on two discrete points is isomorphic to the algebra of pure quantum states with spin-1/2. In addition, if the bound states are not identical the $SU(2)$ symmetry will be broken. This is not an important result, but it does give us reason to look for deeper coincidences in the bound state algebra and the standard model fermions.

8 Application: Koide's Mass Formula

In the standard model, the elementary fermions are composed of left and right handed parts. The parts interact with each other through a massive scalar particle(s) the Higgs boson. The mass of the higgs is around 1 to 2×10^{11} eV. This is quite heavy. The lightest of the charged leptons is the electron at 5×10^5 eV and the heaviest is the tau at 1.7×10^9 eV. And the neutral leptons, the neutrinos, weigh a fraction of an eV. It's interesting to note that the leptons, both charged and neutral, have masses far less than that of the particle that gives them mass.

One of the subjects we will discuss here is the masses of the leptons, mesons and baryons. In each of these cases we will assume that the observed masses are entirely due to interactions between the particles rather than inherent to the particles themselves. For this sort of situation, it's natural to try to use perturbation theory to find a mass formula. Normally one would do this with quantum field theory but the objective of this paper is to look for models that use only quantum mechanics. Therefore, we will now work out a formula for the masses of the charged leptons from basic quantum mechanics perturbation theory. The same theory will apply to the mesons and baryons which we will discuss later.

To make a perturbation calculation, we'll suppose that the leptons and some unspecified higher mass particles (perhaps Higgs) are all manifestations of the same quantum state, but in different excitations. Let $H^{(0)}$ be the first order energy operator for the leptons and some unspecified, under which the Lepton energies (and masses) are all zero and the other particles have very high energies. Write the first order energies of the leptons as $E_n^{(0)} = 0$. Assume a perturbation correction to the Hamiltonian $H^{(1)}$. One finds[6] that the first three terms in the energy eigenvalue perturbation series consists of:

$$E_n = E_n^{(0)} + \langle \psi_n^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle + \sum_k \frac{|\langle \psi_k^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}} \quad (53)$$

where $\psi_n^{(0)}, E_n^{(0)}$ are the first order wave functions and energies for the leptons, $\psi_k^{(0)}, E_k^{(0)}$ are the same but for the high energy particles that give the leptons mass.

Since the leptons are assumed massless to first order, and since their masses are to come from an interaction with a higher energy particle, we assume that the first two terms of the above series are zero and that the

leading nonzero contribution is the third term:

$$E_n = -\sum_k \frac{|\langle \psi_k^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle|^2}{E_k^{(0)}}. \quad (54)$$

Furthermore, it's natural to suppose that the above term is dominated by the contribution from one particular k . This gives us a simple formula for the lepton masses; they might be written as the square of a matrix element between the lepton and some high energy excitation:

$$E_n = -|\langle \psi_k^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle|^2 / E_k. \quad (55)$$

In pursuing this idea further, we should look around for relationships among the square roots of the lepton masses.

Another reason for looking at square roots of masses is that when we look at the energy contained in a field that satisfies linear superposition, we find that the energy is proportional to the squared magnitude of the field. For example, the energy in the electromagnetic field is:

$$m/c^2 = E = \int (|\mathbf{E}|^2 + |\mathbf{B}|^2) d^3x. \quad (56)$$

We don't mean to suggest the electromagnetic field as the source for particles masses, but instead to bring the subject up to discuss how mass might arise from a field. As with the 2nd order perturbation equation of Eq. (54), mass is a sum of squares.

While the formula for the energy in the electromagnetic field treats the electric field and magnetic field equivalently, it is unlikely that their contributions to the mass of a particle would be equal. It's more likely that one would dominate. If so, taking the square roots of masses would be a way of obtaining quantities proportional to a field making up the particle. Since it is the field that satisfies linear superposition, not the energy, linear formulas relating the square roots of masses are not unexpected.

For the case of electromagnetism, it's known that if magnetic monopoles exist, the mass of such a particles is much larger than the mass of the corresponding electric monopoles (i.e. the electron). Thus we might expect that only the electric field would be present and so taking the square root of the mass would give the electric field. Or alternatively, both fields might be present but in an energy ratio suitable for their respective monopoles. In that case, the magnetic field energy would dominate the energy, and the square root of the mass would be a linear approximation of the strength of the magnetic field.

So there are several reasons for looking at the square roots of the masses of elementary particles. Of the particles that are known, the elementary ones whose masses are most accurately known are the charged leptons, the electron, the muon, and the tau. In 1980, Yoshio Koide noticed that the masses of the charged leptons approximately satisfy the relation:

$$2(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 3(m_e + m_\mu + m_\tau). \quad (57)$$

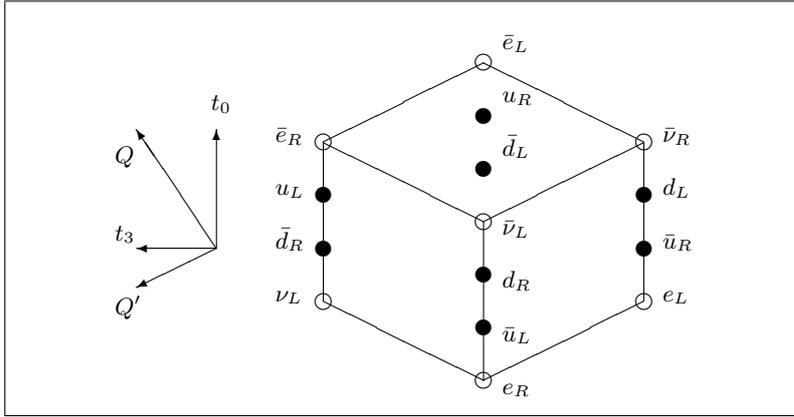


Fig. 2 Weak hypercharge, t_0 , and weak isospin, t_3 , quantum numbers plotted for the first generation standard model quantum states. Leptons are hollow circles and quarks ($\times 3$) are filled circles. Electric charge, Q , and neutral charge, Q' also shown.

The above is a quadratic equation in the square roots of the masses. It can be solved to give the mass of the tau in terms of the masses of the electron and muon. Thus Koide's formula, written at a time when the tau mass had not been accurately measured, gave a prediction for the tau mass. It turned out to be quite a good prediction; 27 years later the predicted tau mass is near the center of the error bars on the latest measurement.

Solving Koide's quadratic equation for the tau mass, and putting in the minimum and maximum electron and muon masses from the Particle Data Group, his prediction for the tau mass is surprisingly close to the center of the error bars:

Lepton	MeV	
e :	0.510998918(44)	
μ :	105.6583692(94)	(58)
τ :	$1776.99 \pm_{-0.26}^{+0.29}$	
Koide τ :	1776.968864(71)	

In 2004, this author began studying the primitive idempotent structure of Clifford algebras. This is a mathematical problem whose solution has been known for years;[7] the short answer is that the primitive idempotents show up in hypercube form with the dimension of the cube depending on the size of the Clifford algebra. So, in the context of Schwinger's measurement algebra or the algebra of bound states, it's natural to see if the quantum numbers of the elementary particles come in hypercube form.

Plotting the handed elementary fermions according to their weak hypercharge and weak isospin quantum numbers, Fig. (2), we see that the 8 leptons fall on the corners of a tilted cube.[3] Each of the four vertical lines of the cube has a lepton at either end, and two quarks at the 1/3 and 2/3 positions on the edge. Particles and antiparticles alternate on the vertical edges. The quarks come in 3 colors so the two (intermediate) quark positions represent 3 particles each. Thus the 2 leptons and 6 quarks come in a $(1, 3, 3, 1)$ pattern

that suggests that the quarks and leptons are each composed of three preons with the leptons pure and the quarks mixtures.

For the leptons to work, the three preons have to be distinguished. This is a problem similar to that of how quarks combine, so we assume that there is a quantum number that distinguishes them, say precolor. Each quark has one preon that is different from the other two; the color of the quark is defined by the color of this preon.

This model of the fermions is quite obvious and probably dates to soon after quarks were postulated. The reasoning against this model was something like this: In order to get Pauli statistics the preons must also be fermions and according to the spin-statistics theorem must a half integer spin, presumably spin-1/2. In combining two distinct particles with spin-1/2 one does end up with 8 states. Unfortunately, the 8 states do not have the correct quantum numbers. The total multiplicities need to be $(1, 3, 3, 1)$, but when one combines three spin-1/2 particles one ends up with a spin-3/2 multiplet (i.e. a $(1, 1, 1, 1)$ set), and two spin-1/2 multiplets, (which will be $(1, 1)$ sets). This is not at all the structure of a column of Fig. (2).

The history of elementary particles has been that new subparticles have always had properties that were surprising. For the present case, the preons would have to be something different from the usual fermions. The solution might have something to do with the unification of gravitation with the other forces. In 2002, Lubos Motl[8] explored the thermodynamics of the quasinormal vibration modes of black holes. He found that the spin-1/2 and spin-1 cases had the expected Fermi and Bose statistics. But the spin-0 case, which would apply to the scalar Higgs particle, had thermodynamics appropriate for a strange sort of Fermi particle. He called it “tripled Pauli statistics” and wrote: “Such an occupation number can be derived for objects that satisfy the Paulis principle, but if such an object does appear (only one of them can be present in a given state), it can appear in three different forms.” Thus one cannot exclude the possibility that preons could require a modification of the usual rules. The preons could be individual spin-0 particles and characteristics of the quarks and leptons, spin-1/2 and the fermion statistics, arises as an interaction between the preons.

Later we will rewrite Koide’s mass equation in a form that puts the masses in the form of the squares of matrix elements. In doing this, we will find another possibly even stranger, coincidence in the masses of the leptons. And then we will apply the same technique to the neutrinos, mesons, and baryons and exhibit coincidences there as well.

9 The Color Problem

A meson is made up of a quark and anit-quark, for example an up quark and an anti-down quark. The quarks come in three colors, R , G , and B . Similarly the anti-quarks come in three anti-colors, \bar{R} , \bar{G} , and \bar{B} . These colors are changed by the exchange of gluons. For a full QFT model, there will be other particles but we will analyze a stripped down model with just these valence quarks. In this model, the color of the quark has to be complementary

to the color of the anti-quark. Thus the only degree of freedom for the system is the color of the quark; the anti-quark carries the corresponding anti-color.

The bound state algebra for the system has 3 degrees of freedom; we must solve the primitive idempotency problem for 3×3 matrices. In putting the problem into matrix form we will use the indices in order R, G, B so the idempotency problem for color transition amplitudes will be written:

$$\begin{pmatrix} a_{RR} & a_{RG} & a_{RB} \\ a_{GR} & a_{GG} & a_{GB} \\ a_{BR} & a_{BG} & a_{BB} \end{pmatrix} = \begin{pmatrix} a_{RR} & a_{RG} & a_{RB} \\ zcv a_{GR} & a_{GG} & a_{GB} \\ a_{BR} & a_{BG} & a_{BB} \end{pmatrix}^2. \quad (59)$$

This gives coupled quadratic equations in nine unknowns. As with the 2×2 problem, there are an infinite number of solutions. Suppose that we have a solution and that it has at least one nonzero off diagonal. Then we will automatically get two more solutions (not necessarily orthogonal) for a total of 3 by modifying the off diagonal elements as follows:

$$\begin{pmatrix} a_{RR} & w^{+n} a_{RG} & w^{-n} a_{RB} \\ w^{-n} a_{GR} & a_{GG} & w^{+n} a_{GB} \\ w^{+n} a_{BR} & w^{-n} a_{BG} & a_{BB} \end{pmatrix} \quad (60)$$

where $n = 1, 2, 3$ or $n = 0, 1, 2$ and w is the complex cubed root of unity: $w = \exp(2i\pi/3)$. This transformation treats the colors equally in that the phase adjustment to a_{RG} is the same as the phase adjustment to the elements given by cyclic permutations of its indices, a_{GB} and a_{BR} . We will require that our solutions be defined according to this transformation, which is equivalent to requiring that the matrices be circulant.

For color, the $SU(3)$ symmetry is unbroken. We therefore require that the colors be treated equally. To ensure this, we require that the magnitudes of the amplitudes be unchanged by colors swaps. The nine squared magnitudes separate into two classes, depending on whether or not they fall on the diagonal. So define non-negative real numbers a and b by:

$$\begin{aligned} |a_{RR}|^2 &= |a_{GG}|^2 = |a_{BB}|^2 = a^2, \quad \text{and} \\ |a_{RG}|^2 &= |a_{RB}|^2 = |a_{GR}|^2 = |a_{GB}|^2 = |a_{BR}|^2 = |a_{BG}|^2 = b^2. \end{aligned} \quad (61)$$

The general matrix is then

$$\begin{pmatrix} ae^{i\theta_{RR}} & be^{i\theta_{RG}} & be^{i\theta_{RB}} \\ be^{i\theta_{GR}} & ae^{i\theta_{GG}} & be^{i\theta_{GB}} \\ be^{i\theta_{BR}} & be^{i\theta_{BG}} & ae^{i\theta_{BB}} \end{pmatrix} \quad (62)$$

where θ_{jk} are real numbers. The requirement that the solutions be primitive gives

$$1 = a(e^{i\theta_{RR}} + e^{i\theta_{GG}} + e^{i\theta_{BB}}), \quad (63)$$

that is, the trace must be unity.

Let us now require that the transformation given in Eq. (60) give three annihilating solutions. We will look at the product of the $n = 0$ solution with itself and the $n = 1$ and $n = 2$ solutions. This would be a lot of algebra so we will write only the equations for the top left corner of the matrices. The

$n = 0$ solution is to give itself when multiplying itself but when it multiplies the other solutions it should give zero:

$$\begin{aligned} ae^{i\theta_{RR}} &= a^2 e^{2i\theta_{RR}} + b^2 (e^{i(\theta_{RG} + \theta_{GR})} + e^{i(\theta_{RB} + \theta_{BR})}), \\ 0 &= a^2 e^{2i\theta_{RR}} + b^2 (e^{i(\theta_{RG} + \theta_{GR} + 2\pi/3)} + e^{i(\theta_{RB} + \theta_{BR} - 2\pi/3)}), \\ 0 &= a^2 e^{2i\theta_{RR}} + b^2 (e^{i(\theta_{RG} + \theta_{GR} + 4\pi/3)} + e^{i(\theta_{RB} + \theta_{BR} - 4\pi/3)}). \end{aligned} \quad (64)$$

Next we sum up these three equations. The left hand side remains $ae^{i\theta_{RR}}$. On the right hand side, we can factor terms from the b^2 terms that sum to zero:

$$1 + e^{+2i\pi/3} + e^{-2i\pi/3} = 1 + (-1/2 + i\sqrt{3}/4) + (-1/2 - i\sqrt{3}/4) = 0. \quad (65)$$

This leaves only the a^2 terms on the right hand side and we have:

$$ae^{i\theta_{RR}} = 3a^2 e^{2i\theta_{RR}}. \quad (66)$$

Thus $a = 1/3$ or and, since the same applies to the other diagonal elements, $\theta_{RR} = \theta_{GG} = \theta_{BB} = 0$.

Now consider the off diagonal idempotency requirement for the $n = 0$ case. We have:

$$be^{i\theta_{GR}} = (2b/3)e^{i\theta_{GR}} + b^2 e^{i(\theta_{GB} + \theta_{BR})}. \quad (67)$$

If $b = 0$ the above is satisfied but then our transformation, Eq. (60), does not give three different solutions so we can assume that $b \neq 0$ and multiply by a phase to get:

$$1/3 e^{i(\theta_{GR} - \theta_{GB} - \theta_{BR})} = b. \quad (68)$$

The right hand side is real so $\theta_{GR} = \theta_{GB} + \theta_{BR} + 2n\pi$ and $b = 1/3$. Putting $a = 1/3$, $b = 1/3$, and $\theta_{RR} = 0$ back into the first equation of Eq. (64), that is, the idempotency equation for a diagonal element, we have:

$$\begin{aligned} 1/3 &= 1/9 + (1/9)(e^{i(\theta_{RG} + \theta_{GR})} + e^{i(\theta_{RB} + \theta_{BR})}), \text{ so} \\ 2 &= e^{i(\theta_{RG} + \theta_{GR})} + e^{i(\theta_{RB} + \theta_{BR})}. \end{aligned} \quad (69)$$

The right hand side can reach 2 only if both terms have their maximum real value, 1. Thus we have $\theta_{RG} = -\theta_{GR}$ and similarly for the other off diagonal phases.

All this leaves us with just three undetermined parameters, θ_{RG} , θ_{GB} , and θ_{BR} . From Eq. (68) they must sum to $2n\pi$, so we are down to two parameters and we can write our 3 primitive bound color states as:

$$\frac{1}{3} \begin{pmatrix} 1 & e^{+i(\theta_{RG} + 2n\pi/3)} & e^{+i(\theta_{RG} + \theta_{GB} - 2n\pi/3)} \\ e^{-i(\theta_{RG} + 2n\pi/3)} & 1 & e^{+i(\theta_{GB} + 2n\pi/3)} \\ e^{-i(\theta_{RG} + \theta_{GB} - 2n\pi/3)} & e^{-i(\theta_{GB} + 2n\pi/3)} & 1 \end{pmatrix}. \quad (70)$$

If the three positions, R , G , and B are identical, we are justified in adding an additional requirement, that the amplitudes be unchanged on cyclic permutations of color so that: $\theta_{RG} = \theta_{GB} = \theta_{BR} =$. Substituting this into the requirement that $\theta_{RG} + \theta_{GB} + \theta_{BR} = 2n\pi$, gives us that these are equal to $2n\pi/3$ for $n = 1, 2, 3$. The three cases amount to the three solutions given by

the transformation of Eq. (60). This gives the general circulant solution to the primitive idempotency problem on 3 positions as:

$$\frac{1}{3} \begin{pmatrix} 1 & e^{+2in\pi/3} & e^{-2in\pi/3} \\ e^{-2in\pi/3} & 1 & e^{+2in\pi/3} \\ e^{+2in\pi/3} & e^{-2in\pi/3} & 1 \end{pmatrix}, \quad (71)$$

which we will connect to the Koide mass formula .

To get a mass, or a field strength, or any other scalar quantity out of the above matrix requires that we select a scaling. The diagonal terms are different from the off diagonal terms in character and may require a different scaling constant. We will use the real number v for the diagonal scaling and a complex number, $s/2e^{i\delta}$ for the off diagonal scaling. The division by two simplifies later equations. Applying this to Eq. (71) the reduction of the three matrices to three scalars is:

$$\begin{aligned} \lambda \left(\frac{1}{3} \begin{pmatrix} 1 & e^{+2in\pi/3} & e^{-2in\pi/3} \\ e^{-2in\pi/3} & 1 & e^{+2in\pi/3} \\ e^{+2in\pi/3} & e^{-2in\pi/3} & 1 \end{pmatrix} \right) &= (3v + 1.5se^{+i\delta} \frac{1}{3} e^{+2in\pi/3} + 1.5se^{-i\delta} e^{-2in\pi/3}) \frac{1}{3}, \\ &= v + s(e^{+i(\delta+2n\pi/3)} + e^{-i(\delta+2n\pi/3)})/2, \\ &= v + s \cos(\delta + 2n\pi/3) = \lambda_n. \end{aligned} \quad (72)$$

The Koide relation is between the square of the sum and the sum of the squares. For the square of the sum we get:

$$(\Sigma_{n=1}^3 \lambda_n)^2 = (\Sigma_n v + s \cos(\delta + 2n\pi/3))^2 = (3v)^2 = 9v^2, \quad (73)$$

where recourse has been taken to the fact that $\Sigma_n \cos(\delta + 2n\pi/3)$ does not depend on δ and instead is always zero. Similarly, $\Sigma_n \cos^2(\delta + 2n\pi/3) = 3/2$ allows us to sum the squares as follows:

$$\begin{aligned} \Sigma_{n=1}^3 \lambda_n^2 &= \Sigma_n (v + s \cos(\delta + 2n\pi/3))^2, \\ &= \Sigma_n (v^2 + 2s \cos(\delta + 2n\pi/3) + s^2 \cos^2(\delta + 2n\pi/3)), \\ &= 3v^2 + 1.5s^2, \end{aligned} \quad (74)$$

and neither the sum of squares nor the square of the sum depends on δ . Koide's relation is that $2(\Sigma_n \lambda_n)^2 = 3\Sigma_n \lambda_n^2$. In terms of s and v , this amounts to:

$$\begin{aligned} 2(9v^2) &= 3(3v^2 + 1.5s^2), \\ s^2/v^2 &= 2. \end{aligned} \quad (75)$$

Thus we can obtain Koide's relation by assuming that $v = \sqrt{2}s$.

The relationship $v = \sqrt{2}s$ is simpler than Koide's relation and one might think of it as more significant because of this but it is still just a single coincidence and this form depends on a lot of manipulation. However, while Koide's formula is a single equation relating three masses, the formula derived here,

$$\lambda_n = v + s \cos(\delta + 2n\pi/3) \quad (76)$$

gives three equations for three unknowns. Of course there has to be an overall scaling, but when Koide's formula is written in this form, there is the possibility of another coincidence, in the angle δ .

Calculating the angle δ is straightforward. Given three masses, m_n , put $\lambda_n = \sqrt{m_n}$. Then

$$\begin{aligned} v &= \Sigma_n \lambda_n / 3, \\ s &= \pm \sqrt{\Sigma_n m_n - 3v^2} / 3, \\ \delta &= \pm \cos^{-1}((\lambda_1 - v)/s), \end{aligned} \quad (77)$$

where the two \pm s are independent. Also note that the value of δ could depend on which of the three masses is chosen as 1. This adds an additional arbitrary phase to δ of $2n\pi/3$. Thus depending on how one chooses the first mass, and how one chooses the other two signs, one has $3 \times 2 \times 2 = 12$ choices for δ .

In the PDG, the lepton masses are given as best, minimum and maximum values. We can use the best values to get nominal values for s , v , and δ , and we can try the various combinations of minimums and maximums to get (slightly over estimated) error bars on these values. Following this procedure, we find:

$$\begin{aligned} v_{cl} &= 17.7160(12) \sqrt{\text{MeV}}, \\ s_{cl} &= 25.0544(22) \sqrt{\text{MeV}}, \\ (s_{cl}^2/v_{cl}^2) &= 2.000007(98), \\ \delta_{cl} &= 0.222221(21) = 12.7323(11) \text{ deg}. \end{aligned} \quad (78)$$

where “ cl ” has been appended to distinguish these numbers, for the charged leptons, from other applications we’ll discuss later. The coincidence that Koide found is that s^2/v^2 is close to 2. The second coincidence is that the angle δ is approximately $2/9$. In both cases, not only are the numbers within the experimental error bars, they are very close to the center, to the best experimental values. The resulting equation for the charged lepton masses is:

$$\sqrt{m_{cl,n}} = (\sqrt{1/2} + \cos(2/9 + 2n\pi/3)) \times 25.0544 \sqrt{\text{MeV}}, \quad (79)$$

is accurate to around 6 decimal places. In analyzing the mesons and baryons, we will keep the overall proportionality constant, $25.0544 \sqrt{\text{MeV}}$, so that any relationships between the leptons and the baryons will be easier to see.

10 Application: The Neutrinos

In this paper, we will define the three generations of neutrinos as the three mass eigenstates labeled ν_1 , ν_2 , and ν_3 . Due to an accident of experiment and the history of particle physics, custom is that the neutrinos are labeled according to the weak interaction as ν_e , ν_μ , and ν_τ . These are not eigenstates of mass, that is, they do not have precise masses as other particles do. Instead, they are eignstates of the weak interaction.

Neutrinos participate only in the weak interactions, they have no electric or color charge. In the weak interactions, when an anti-neutrino is produced, the same interaction will produce an electron, a muon, or a tau. The reverse applies to neutrinos, when they are produced, a positron (anti-electron), an anti-muon, or an anti-tau is produced. Just considering the leptons, the weak interactions convert a charged lepton into a neutral lepton or vice versa. The quarks also participate in the weak interactions but we will not discuss them here.

Various other interactions are possible, for example, if a neutrino is absorbed this implies a charged lepton (rather than a charged anti-lepton) is created, and it is possible for an anti-neutrino to annihilate itself with an electron or other charged lepton. The heavier charged leptons can decay into lighter leptons along with a neutrino and an anti-neutrino. For example, a tau can decay into a tau-neutrino, an electron, and an anti-electron neutrino.

In the original standard model, the neutrinos were assumed to be massless. This implied that the type of charged lepton (or anti-lepton) produced with them would be a good quantum number for the neutrino. Accordingly, the neutrinos were named the electron neutrino, muon neutrino, and tau neutrino with symbols ν_e , ν_μ , and ν_τ . Suppose a tau neutrino decayed into an electron and in the process emitted a tau neutrino. Under the original standard model, that tau neutrino could convert an electron into a tau only, never a muon. Experimentalists eventually found that neutrinos were not so simple, they could change type in flight, but only in long flight. When a beam of pure ν_τ is created, it eventually becomes a mixture of all three neutrino types. This mixing is known as neutrino oscillation.

Students will understand neutrinos much more easily if they are treated in the same way as all other particles are, that is, as the mass eigenstates ν_1 , ν_2 , and ν_3 . When a charged lepton decays, rather than emitting just one type of neutrino as was assumed in the original standard model, the decay can proceed with the emission of any of these three. For a given charged lepton, there are three different ways it can decay and therefore three different amplitudes. These amplitudes are given by the so-called “neutrino mixing matrix” or ABC^2 matrix. There is an analogous mixing in the quarks, which uses a different matrix, the CKM matrix.

When one uses Feynman diagrams to compute a probability, one first writes down all the possible interactions (as Feynman diagrams). Each individual diagram is then converted into a complex number, its amplitude. All the amplitudes are then added up. The probability is the squared magnitude of this sum of individual amplitudes.

In the case of a beam of charged leptons that decay and produce neutrinos, we must include diagrams for all three types of neutrinos. Each of these diagrams produces its own individual amplitudes. These amplitudes are solutions to the wave equation for a fermion particle, that is, they are solutions to the Dirac equation:

$$(\gamma^\alpha \partial_\alpha + m_{\nu k})\psi = 0, \quad (80)$$

where $m_{\nu k}$ is the mass of the k th neutrino.

Since the three different neutrinos have three different masses, for any given energy they have three different frequencies and three different wavelengths. Consequently, their amplitudes will interfere with each other over distance. This has the effect of increasing or decreasing the probability of a weak interaction of a particular type. For example, if the neutrinos come from a muon decaying to an electron, then it will produce the ν_1 , ν_2 , and ν_3 with a certain relative phase appropriate for the muon (and it will produce three anti-neutrinos with relative phase appropriate for an anti-electron). In

² fix this

the immediate neighborhood these relative phases will be unchanged. The phases appropriate for the muon will be more likely to cause an electron to transform into a muon. But as the distance increases, the relative phases of the neutrinos changes and this decreases the probability of creating a muon.

Since the neutrino masses are so small, and since the particles have no electric charge, it is impossible to measure their masses in the traditional way. In fact, at this time, no accurate neutrino masses are known. On the other hand, the effect of neutrino interference (or oscillation) is the subject of much experimental effort and the numbers for neutrino interference are now known to between 1 and 2 significant digits.

If two neutrinos had identical masses, then their Dirac equation solutions would be identical and there would be no interference. Consequently, neutrino interference experiments provide a measurement of how different two neutrino masses are. The measurements give the difference between the squares of the masses. Furthermore, they cannot tell which mass is larger:

$$\begin{aligned} |m_{\nu 2}^2 - m_{\nu 1}^2| &= 7.92(1 \pm .09) \times 10^{-5} \text{ eV}, \\ |m_{\nu 3}^2 - m_{\nu 2}^2| &= 2.4(1 + 0.21 - 0.26) \times 10^{-3} \text{ eV}. \end{aligned} \quad (81)$$

The two different mass measurements are far enough apart that there is a hint of the same sort of generation structure seen in the charged leptons.

Given the excellent agreement of Koide's equation for the charged leptons, a natural instinct is to see if Koide's formula, Eq. (57), can be used to derive the neutrino masses from these two oscillation measurements. Several papers in the literature stated that this cannot be done.[9] [10] This depended on the assumption that the square roots in Koide's formula must be taken to be positive. The author corrected this oversight in 2005.

Ignoring the overall mass scale, three masses that satisfy Koide's formula can be put into the form of Eq. (76) that automatically satisfies Koide's equation. This leaves only one parameter left, δ :

$$\lambda_n(\delta) = \sqrt{1/2} + \cos(\delta + 2n\pi/3). \quad (82)$$

The cosine can be taken as the real part of the complex phase $\exp(i(\delta + 2n\pi/3))$ which we can draw in the complex plane. The three square roots of mass appear as three points equidistant on a unit circle offset from the origin by $\sqrt{1/2}$. The square root of the mass is given by the real part of the complex number. Since $1 > \sqrt{1/2}$, it is possible for one of the three particles to have a negative square root mass. This happens with the neutrinos, as shown in Fig. (3).

Adding the Koide relation to the oscillation data gives the following neutrino masses:

$$\begin{aligned} m_{\nu 1} &= 0.000388(46) \text{ eV}, \\ m_{\nu 2} &= 0.00895(17) \text{ eV}, \\ m_{\nu 3} &= 0.0507(30) \text{ eV}. \end{aligned} \quad (83)$$

These satisfy the Koide relation as

$$3(m_{\nu 1} + m_{\nu 2} + m_{\nu 3}) = 2(-\sqrt{m_{\nu 1}} + \sqrt{m_{\nu 2}} + \sqrt{m_{\nu 3}})^2. \quad (84)$$

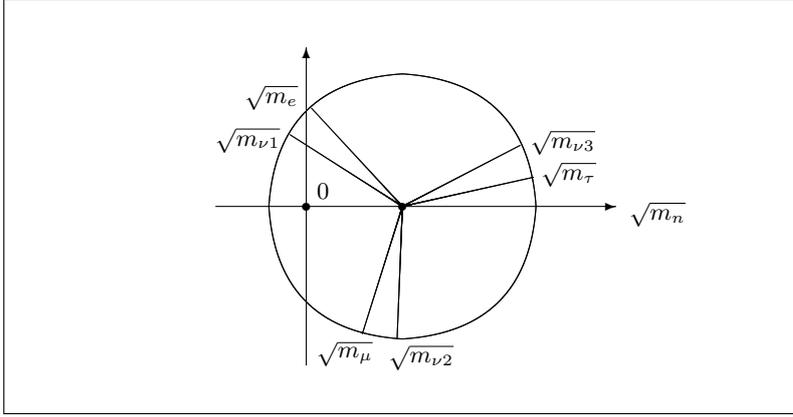


Fig. 3 Three masses related by Koide's formula can be drawn as three equidistant points on a circle. The real axis gives the square roots of the masses. This figure shows the charged leptons and neutrino masses.

Putting these into $v + s \cos(\delta + 2n\pi/3)$ form, we have:

$$\sqrt{m_{\nu n}} = (\sqrt{0.5} + \cos(0.486(21) + 2n\pi/3)) \times (5.64(15) \times 10^{-6}) \times 25.0544 \sqrt{\text{MeV}} \quad (85)$$

The constant 5.64×10^{-6} defines the ratio of the field strengths of the neutrino and the electron. This constant is very close to 3^{-11} . In perturbation theory, different diagrams take different powers of the coupling constant or charge. Seeing a power of 3 suggests that the preons making up the electron and neutrino may have a coupling constant of $1/3$ and the explanation for the very light masses of the neutrinos is that the leading order Feynman diagram for their mass interaction has eleven extra vertices.

The angle δ for the neutrinos, $0.486(21)$ is interesting in that this differs from the angle for the electrons by approximately $\pi/12$. We have

$$0.486(21) - 2/9 = 0.264(21) = 1.008(80) \frac{\pi}{12}. \quad (86)$$

Because of the $2n\pi/3$ inside the cosine, it is always possible to modify δ by adding $2n\pi/3$ to it, this could be used to modify the $\pi/12$ to $3\pi/4$. Instead, we will attribute the $\pi/12$ angle to be a result of Berry-Pancharatnam phase on the complete set of MUBs of the Pauli algebra. We will discuss this further in Sec. (12). The mathematics for this is somewhat difficult and so, to increase the motivation for understanding the angle $2/9$, we will apply the bound state algebra on 3 colors to the mesons.

11 Application: The Mesons

At its simplest, a meson consists of a quark and an anti-quark, bound together by the color force, for example, the K^0 meson is composed of a $s\bar{d}$. There

are several hundred mesons listed in the Particle Data Group's 2006 records; organizing them according to the principle of the bound state algebra requires some effort. In addition, different mesons have different levels of certainty; some could be the result of fluctuations or experimental error. So we will begin with a brief introduction to the nomenclature of mesons. For this first part, we will assume that mesons have specific quark content rather than linear superpositions of quarks. We will cover the, more complicated, linear superposition cases at the end of the section.

The standard meson names consist of a symbol, such as K for the K -mesons. This symbol defines the quantum numbers for the state other than its total angular momentum (and sometimes isospin and therefore electric charge). Mesons that have up or down quark (u or d) content come in isospin multiplets. In an isospin multiplet, charge (and therefore up and down quark content) is indicated by a superscript, $+$ for each multiple of positive charge, $-$ for each multiple of negative charge, and 0 if there is no electric charge. These multiplets arise because the up quark, and the down quark, are quite nearly interchangeable though its mass changes slightly and its electric charge changes by 1 . For example, the $K^0 = s\bar{d}$ meson, which has charge $-1/3 - (-1/3) = 0$, comes with the $K^- = s\bar{u}$ meson which has charge $-1/3 - (2/3) = -1$.

In addition to isospin multiplets, mesons also have anti-particles partners. To convert a meson to its anti-particle, one swaps the quark for the anti-quark. Thus the antiparticle for the $K^0 = s\bar{d}$ is the $\bar{K}^0 = d\bar{s}$. If the quark and anti-quark are the same, then the meson is its own anti-particle and there is no notation for the anti-particle. The anti-particle partner to a quark has the opposite electric charge so the anti-particle partner to a meson also has the opposite electric charge. So, if a meson is charged, there is no reason to indicate the anti-particle with a bar and one instead indicates which is which by its electric charge. Thus the $\pi^+ = u\bar{d}$ and the $\pi^- = d\bar{u}$ are anti-particles. It should be noted that there is no absolute way to choose which meson is the "particle" and which is the "anti-particle". Among the mesons, and among elementary particles in general, particle and anti-particle is a relation between pairs of particles. It is not a measured quantity like electric charge where one can say that all the particles with charge $+1$ are alike. Historically, physicists have applied the "particle" designation to the quantum state which is more common in nature or discovered first.

Excitations of a meson exist. These have quantum numbers identical to the meson but have different energies. These mesons are designated by giving the approximate mass (MeV) in parentheses after the symbol. For example, the $\pi^+(1300)$ is an excited state of the π^+ meson with a mass of approximately 1300 MeV. Along with the $\pi^+(1300)$ meson there will be the isospin partners with approximately the same mass, that is, the $\pi^0(1300)$ and $\pi^-(1300)$. Because the isospin partners have attributes that are predictable relative to each other, and since their small differences in mass are difficult to measure, one generally refers to the multiplet together. Thus the Particle Data Group will have an entry for $\pi(1300)$ rather than separate entries for the three isospin partners.

One sometimes one finds a meson that has the same quantum numbers as one seen before except for its total angular momentum J . These mesons are designated by giving the total angular momentum as a subscript to the symbol. These are angular excitations. For example, the $\pi^+ = u/\bar{d}$ has $J = 0$ while the $\pi_1^+(1400)$ is also composed of $u\bar{d}$ but has $J = 1$. And these come in isospin multiplets that are abbreviated as $\pi_1(1400)$.

Ignoring isospin multiplets, the complete list of π mesons in the Particle Data Group files is:

$$\begin{array}{cccc}
 \pi(138) & \pi_1(1400) & \pi_2(1670) & \pi_4(2250)^* \\
 \pi(1300) & \pi_1(1600) & \pi_2(1880)^* & \\
 \pi(1800) & \pi_1(2015)^* & \pi_2(2005)^* & (87) \\
 \pi(2070)^* & & \pi_2(2100)^* & \\
 \pi(2360)^* & & \pi_2(2245)^* &
 \end{array}$$

where “ $\pi(138)$ ” is the usual π^+ or π^0 , and the asterisks indicate states that are not as well established experimentally (or are even suspected of not existing) and therefore are “omitted from summary table.” Each of the above stands for an isospin vector of three mesons. In addition to the above pions, the other up and down quark mesons are the η , f , ρ , ω , a , ϕ , h , and b . The most prolific of these is the f with twice as many states as the above π . And as one adds in mesons with the strange, charm, bottom, and top quarks, the number of meson states increase, however, the bulk of the known or claimed meson states are made from just the up and down quarks.

In the meson literature, one sometimes sees the phrase “radial excitation” to refer to excitations of mesons like those in the 2nd through 5th lines of the above table. Since mesons come in different total angular momentum, it is clear that radial excitations do exist. However, we claim that not all the states are radial excitations, some are color excitations as seen in Sec. (9). We will be applying the Koide formula to these triplets of states, that is, we will write the three masses as $\sqrt{m_n} = v + s \cos(\delta + 2n\pi/3)$. A difficulty in doing this is the problem of assigning mesons to the right Koide-triplet. That is, given a set of six mesons with the same quantum numbers, we have no obvious way to separate them into two groups of three (other than trial and error).

Among the hydrogen wave functions, radial excitations have energies of $1/n^2$. This makes the energies much closer together as n increases. The difference in energy between $1/1^2$ and $1/2^2 = 1/4$ is $3/4$, while the difference in energy between $1/2^2$ and $1/3^2$ is $5/36$. Thus for radial excitations, we expect consecutive energy differences to be $(3/4)/(5/36) = 5.4$. For the first three π resonances, the actual energy differences are $(1300 - 138)/(1800 - 1300) = 2.3$ which is considerably smaller; our claim is that the three lowest lying pion states are color excitations with approximately equivalent radial wave functions.

The radial excitation effect is seen in the mesons in that the total number of states increases drastically as the energy increases, and no mesons at all exist above. Therefore, to test Koide’s formula we will use low energy mesons. Most of these are composed of up and down quarks. Once we see that the

formula works for these simple cases, we will then use the formula to classify the higher energy states.

One sees the phrase “radial excitation” used to explain mesons with identical quantum numbers. Our claim here is that while

One of the most deeply studied high energy mesons is the $J\psi$. There are a total of 6 of these particles. Because of historical precedence, their names do not follow the meson

12 Application: The Delta Angle

In the standard model, the lepton masses appear as arbitrary constants determined by experiment. But in 1982, Yoshio Koide proposed a formula for the charged lepton masses that is still going strong a quarter century later. The success of Koide's formula remains unexplained, but its perfect accuracy, and its simplicity in explaining the charged lepton mass hierarchy, suggest that it may be the basis for a new theory of mass, a theory simpler than that of the standard model. In this paper, we extend the Koide mass formula to an eigenvector equation, find further coincidences, apply the formula to the neutrinos, and speculatively suggest a complete solution to the problem of the hierarchy of lepton masses and the MNS mixing matrix.

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13 Application: Standard Model Particles

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14 Preon Statistics

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15 Application: The Baryons

In the standard model, the lepton masses appear as arbitrary constants determined by experiment. But in 1982, Yoshio Koide proposed a formula for the charged lepton masses that is still going strong a quarter century later. The success of Koide's formula remains unexplained, but its perfect accuracy, and its simplicity in explaining the charged lepton mass hierarchy, suggest that it may be the basis for a new theory of mass, a theory simpler than that of the standard model. In this paper, we extend the Koide mass formula to an eigenvector equation, find further coincidences, apply the formula to the neutrinos, and speculatively suggest a complete solution to the problem of the hierarchy of lepton masses and the MNS mixing matrix.

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16 Conclusion

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