Lepton Masses as Star of David

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Challenged to produce a holiday LaTex on a physics website, we choose to graph the square roots of the masses of the leptons on a circle, and show that they form a Star of David. Some of the leptons have very precisely known masses, particularly the electron and muon. For the neutrinos, we have only the absolute values of the differences between their squares.

Since the neutrino masses are not known precisely, we have a certain freedom in choosing their values. Of course we choose values that are consistent with the Star of David. Since the neutrino masses are very small compared to that of the charged leptons, we multiply the neutrino masses by the interesting constant 3^{11}. Finally, we choose the square root of the lowest mass neutrino to be negative.

With these assumptions, the lepton masses plot on a circle with two equilateral triangles, reminiscent of the Star of David.

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The source paper for this paper is [1], a paper which rewrote the Koide [2–4] mass formula:

$$\frac{(\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau})^2}{m_e+m_\mu+m_\tau}=\frac{3}{2}, \quad (1)$$

in eigenvector form, and generalized it from the charged leptons to all the leptons. That the Koide formula could be fit to the neutrinos was a surprise as this could not be done [5–7] when the Koide formula was written in the usual form.

Referring to [1], the predicted masses of the charged leptons are:

$$\begin{aligned}
m_e &= \left(\mu_1(1+\sqrt{2}\cos(\delta_1+2\pi/3))\right)^2,\\
m_\mu &= \left(\mu_1(1+\sqrt{2}\cos(\delta_1+4\pi/3))\right)^2,\\
m_\tau &= \left(\mu_1(1+\sqrt{2}\cos(\delta_1+6\pi/3))\right)^2,\\
m_{\nu1} &= \left(3^{-11}\mu_1(1+\sqrt{2}\cos(\delta_1+2\pi/3+\pi/12))\right)^2,\\
m_{\nu2} &= \left(3^{-11}\mu_1(1+\sqrt{2}\cos(\delta_1+4\pi/3+\pi/12))\right)^2,\\
m_{\nu3} &= \left(3^{-11}\mu_1(1+\sqrt{2}\cos(\delta_1+6\pi/3+\pi/12))\right)^2,
\end{aligned} \quad (2)$$

where μ_1 and δ_1 are constants determined by measurements of the electron and muon masses. Recent measurements give:

$$\begin{aligned}
\mu_1 &= 17715.99225(79) \quad \mathrm{eV}^{0.5},\\
\delta_1 &= 0.22222204715(312),
\end{aligned} \quad (3)$$

Note that the μ_1 has units of square root mass, while δ_1 is a pure number. It is particularly interesting that δ_1 is close to $2/9$.

The above formulas include a cosine function. To add an extra dimension to the plots, we replace cos() with $\cos() + i\sin()$, and graph the square roots of the masses on the complex plane. The angles of $2n\pi/3$ give multiples of 120 degrees and that is just what we need to give the equilateral triangles of the Star of David.

In a true Star of David, the two equilateral triangles are offset by 60 degrees. In the above, they are offset by only 15 degrees (i.e. $\pi/12$), and this will give a very distorted star. However, since the cosine is an even function, we can put a minus sign on its argument. This gives an angle between the two equilateral triangles of $(2/9-(-2/9-\pi/12))\times 180/\pi = 40.465$ degrees, which is close enough to 60 degrees that the result will look at least a little bit like a Star of David.

An alternative would be to rotate the axes of the neutrino masses by 45 degrees from the axes of the charged lepton masses. This would give a perfect Star of David, but ugly axes. Oh what to do, what to do. This is getting old, I'll just do the distorted figure with clean mass axes, though the other case is more physical.

Using x for the real part, and y for the imaginary part, the (x, y) coordinates of the lepton masses (with shared mass axis) are:

$$\begin{aligned}
\sqrt{m_e}/\mu_1 &= (+0.040350090, +1.038783929),\\
\sqrt{m_\mu}/\mu_1 &= (+0.580211684, -1.350473165),\\
\sqrt{m_\tau}/\mu_1 &= (+2.379438226, +0.311689236),\\
\sqrt{m_{\nu1}}/\mu_1 &= (-0.195807697, -0.755012551)\times 3^{-11},\\
\sqrt{m_{\nu2}}/\mu_1 &= (+0.944043799, +1.413106119)\times 3^{-11},\\
\sqrt{m_{\nu3}}/\mu_1 &= (+2.251763898, -0.658093568)\times 3^{-11}.
\end{aligned} \quad (4)$$

In the above, the x coordinate gives the square roots of the masses.

The predicted values are within current experimental error as we now show. Applying the measured values for

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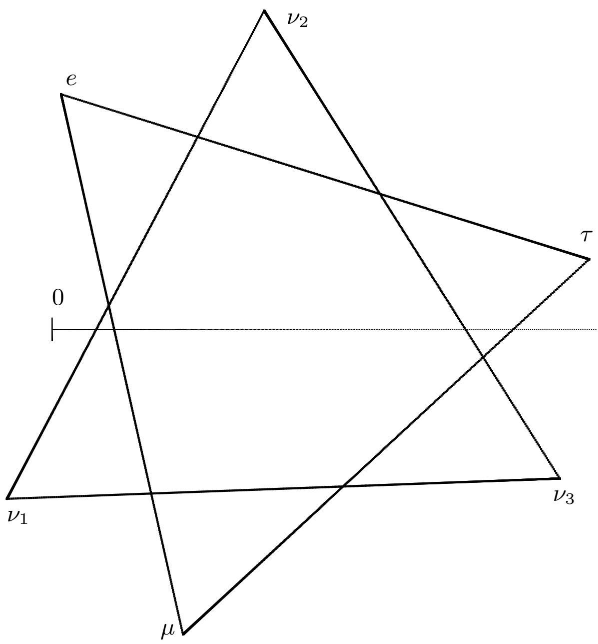

FIG. 1: Lepton masses in Star of David form. Masses are given by square of horizontal coordinate with zero as marked, and neutrino masses multiplied by 3^{-22}. The two triangles are equilateral, are the same size, and share the same center.

μ_1 and δ_1, the predicted masses of the six leptons:

$$\begin{array}{rcll} m_e &=& 0.510998918(44) & \text{MeV}, \\ m_\mu &=& 105.6583692(94) & \text{MeV}, \\ m_\tau &=& 1776.968921(158) & \text{MeV}, \\ m_{\nu 1} &=& 0.000383462480(38) & \text{eV}, \\ m_{\nu 2} &=& 0.00891348724(79) & \text{eV}, \\ m_{\nu 3} &=& 0.0507118044(45) & \text{eV}. \end{array} \quad (5)$$

The latest PDG values for the electron, muon, and tau are:

$$\begin{array}{rcll} m_e &=& 0.510998918(44) & \text{MeV}, \\ m_\mu &=& 105.6583692(94) & \text{MeV}, \\ m_\tau &=& 1776.99(+29-26) & \text{MeV}, \end{array} \quad (6)$$

Since μ_1 and δ_1 are computed from the measured electron and muon masses, they naturally fall in the center of the measured values. The prediction of the tau mass is the same as that of the Koide formula, which, as mentioned above, dates to 1982, long before the tau was accurately measured. And after 24 years, Koide's prediction is still very close to the center of the measurement.

At this time, the neutrino masses are only accurately measured by oscillation experiments, and these measurements give two differences between squares of neutrino masses. Therefore the best we can do in comparison with experiment is to compare the predictions of the differences between the squared magnitudes:

$$\begin{array}{rcll} m_{\nu 2}^2 - m_{\nu 1}^2 &=& 7.930321129(141) \times 10^{-5} & \text{eV}^2, \\ m_{\nu 3}^2 - m_{\nu 2}^2 &=& 2.49223685(44) \times 10^{-3} & \text{eV}^2, \end{array} \quad (7)$$

with the latest oscillation data:

$$\begin{array}{rcll} m_{\nu 2}^2 - m_{\nu 1}^2 &=& 7.92(1 \pm .09) \times 10^{-5} & \text{eV}^2, \\ m_{\nu 3}^2 - m_{\nu 2}^2 &=& 2.4(1 + .21 - .26) \times 10^{-3} & \text{eV}^2. \end{array} \quad (8)$$

Again, the predicted values are near the centers of the measurement bars.

Note: The most accurate measurements of the electron and muon are done not in unites of eV, but instead are in the less familiar AMU. We will use the eV values because of their familiarity to the reader. For the corresponding AMU numbers, see equations (17) and (18) of [1].

I. CONCLUSION

What numerological magic has gone on here? The value $\pi/12$ is difficult to explain to people who have not followed the author's efforts in modeling the elementary fermions in a density operator formalism based on geometry. The standard model fermions are quite complicated particles, and the derivation of their structure from Clifford / geometric algebra must also be fairly difficult.

I have no doubt that very few of the readers of this paper have the patience needed to learn the changes to the foundations of physics that are behind these formulas. Most physicists are satisfied with the current situation, where the foundations are written in symmetry principles. For those who wish to get a hint as to what is going on, I've included in the appendix a brief discussion.

APPENDIX: GEOMETRY AS THE FOUNDATION OF PHYSICS

The old and new foundations of physics mostly share the same mathematical objects. Where they differ is in their interpretation. For example, if you ask almost any physicist what happens when an electron is rotated 360 degrees, they will tell you that the electron is multiplied by -1. This is because the spinor representation of fermions is so pervasive in the standard model that it leads people to confuse the mathematical representation with the physical object. In fact, the density matrix representation of the electron represents the electron as fully as the spinor representation but the density matrix representation is not multiplied by -1 when so rotated.

In the new foundation, we treat the density operator[1] as the fundamental object, and the spinor as a mathematical convenience. The old foundation defines the density matrix from the spinor, while the new foundation

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[1] The designation "density operator" is used in preference to "density matrix" as the new formalism is written in terms of the geometric or Clifford algebra elements, for example, "σ_z" instead of any particular choice of matrix representation such as the Pauli spin matrices.

reverses this relation, and defines the spinor representation from the density operator representation. The fact that this is even possible is almost unknown in the physics community, however the method involves adding a fictitious vacuum state and dates to the 1950s (see Julian Schwinger [8] or the author's text book on applications of the density operator formalism to the standard model at www.DensityMatrix.com).

Physics is very tightly woven together with very long threads. It is not possible to rewrite just a small part of the foundations. After making one change, one finds that one has undermined the foundation in another spot, and one is led to another change. This process continues until one finds that one has completely rewritten the foundations.

As with the issue of spinors versus density matrices, the mathematical objects of the old foundations and the new foundations are mostly identical. Where they differ is in their interpretation. A mathematical object (spinor) that was once so closely associated with the physical object that they can be confused now becomes just a mathematical convenience, a fiction that is convenient for making calculations. And a mathematical object (density matrix) that was assumed to be just a mathematical convenience for calculation now becomes a fundamental part of the physics, that is, becomes a part of the foundations of physics.

The changes to the foundations are numerous and a complete explanation requires a textbook length monograph. The symmetries of physics can be conveniently divided into two groups, discrete and continuous. The continuous symmetries are associated with relativity, while the discrete symmetries mostly appear in elementary particles and quantum mechanics. Accordingly, the author is writing two textbooks, one for the continuous symmetries and one for the discrete.

Since elementary particle theory is deeply built on an assumption of relativity, logically, the continuous symmetries should be discussed first. But the sociology of physics is such that discussing modifications to quantum mechanics is easier than discussing modifications to relativity. In addition, the extensions to the standard model that allow such useful calculations as the lepton masses need the discrete symmetries more than the continuous ones. Accordingly, the author is producing the discrete symmetry textbook first.

Since the foundations of physics are so tightly interwoven, it is very easy for a physicist, when exposed to the new foundations, to find a contradiction. It is very natural to stop as soon as one finds that first contradiction. Each such contradiction requires yet another change to the foundations. One finds that one is wandering farther and farther from the familiar. It requires great faith[2] to forge ahead.

[2] or ignorance

In order to give the reader a reason to doubt the usual foundations, we now include enough theory to show how linear superposition can be accomplished within the density operator formalism, without any use of spinors. One can take a linear superposition of two spinors, and the result will still be a spinor (or zero). The spinors correspond to pure density operators, and when one takes a linear superposition of two pure density operators one generally obtains an impure state, that is, a statistical mixture.

What we will do is show how one can take two pure density operator states, and combine them in a manner similar to the spinor linear superposition, to get a new pure density operator state. We will work within the discrete degrees of freedom for spin-1/2, that is, in the old formalism, our quantum states would consist of all the vectors composed of two complex numbers.

Let ρ_0 be an arbitrary density operator state, we will call it the vacuum state.[3] It is fictitious, largely arbitrary, and we will use it only for mathematical convenience in calculation. Our density operator states satisfy the idempotency equation:

$$\rho^2 = \rho, \tag{A.1}$$

and this applies to the vacuum state as well.

Let ρ_A and ρ_B be two arbitrary density operator states with the usual normalization, that is, $\rho_A^2 = \rho_A$, and $\rho_B^2 = \rho_B$. We require only that they do not annihilate the density operator state, that is, we require that $\rho_A\ \rho_0 \neq 0$ and $\rho_B\ \rho_0 \neq 0$. Given two complex numbers, a and b, we define the linear superposition as:

$$\rho_{aA+bB} = (a\rho_A + b\rho_B)\rho_0(a\rho_A + b\rho_B). \tag{A.2}$$

As with the linear superpositions of spinors, the above is not normalized.

We now compute ρ_{aA+bB}^2. To do this, let us take advantage of the spinor formalism, and choose a spinor $|0\rangle$ that gives $\rho_0 = |0\rangle\langle 0|$. Then

$$\begin{aligned} \rho_{aA+bB}^2 &= (a\rho_A + b\rho_B)|0\rangle\langle 0|(a\rho_A + b\rho_B)^2, \\ &= (a\rho_A + b\rho_B)|0\rangle\ (\langle 0|(a\rho_A + b\rho_B)^2|0\rangle) \\ &\qquad \langle 0|(a\rho_A + b\rho_B), \\ &= \langle 0|(a\rho_A + b\rho_B)^2|0\rangle\ \rho_{aA+bB}. \end{aligned} \tag{A.3}$$

Since $\langle 0|(a\rho_A + b\rho_B)^2|0\rangle$ is a complex number, we see that ρ_{aA+bB} is, indeed, the complex multiple of a density operator (possibly zero).

Note that in the spinor theory, there is an arbitrary choice of complex phase in the spinors $|A\rangle$ and $|B\rangle$. Given any particular complex values a and b, the phase choice will change the linear superposition. In other words, linear superposition is subject to a gauge freedom. In the

[3] The use of "vacuum" is due to analogy with field theory. The terminology is from Schwinger, [8].

density operator formalism, this gauge freedom is made into the choice of the vacuum ρ_0. Since the vacuum state is a pure spin-1/2 state, it defines a direction in space. Therefore, the arbitrary choice of phase in the spinors, becomes, in the density operator formalism, a geometric choice of the fictitious vacuum. The reader is invited to show that ρ_{aA+bB} can be obtained from spinors by appropriate choices of the complex phases of $|A\rangle$ and $|B\rangle$, and that when $a = 0$ or $b = 0$, the result is analogous to the spinor linear superposition.

The very arbitrary and confusing gauge issues of the old foundation are replaced with an elegant and manifest choice of the vacuum state in the new foundation. Also note that while the old foundation requires two different sorts of objects, states and operators, the new foundation requires only operators. And finally, note that the mathematical objects of the two foundations are the same, what has changed is the way they are used and their interpretations.

However, the demonstration is very incomplete. We need to expand this to cover quantum states that are spread over position. And even ignoring that issue, the interrelatedness of the foundation assumptions of physics leave us with further problems also requiring resolution: The method required adding an arbitrary "vacuum" state, and the use of this state suggests that the Higgs vevs of the standard model are unphysical. In addition, the density operators have no arbitrary complex phase, and this suggests that we will have to rewrite the principles of gauge theory.

These are problems that are endemic in rewriting the foundations of physics. In making a few small changes to the old foundations, we generate more problems that can only be solved by making even more changes to more parts of the old foundations and these changes make yet more problems to fix. This process continues, like a phase change, through the foundations of physics until all of the old foundations are replaced with new.

While the first foundation changes one tries must cause more new problems than it solves, one eventually reaches the point where the changes one makes to the foundations are self reinforcing, and solve more problems than they create. The last few modifications fit together with the inevitability of the last pieces of a jigsaw puzzle, and one finds that one has a complete new foundation that is as internally consistent as the old one, but that allows simpler, faster, and more general calculations than the old. The sociological problem is that showing it to someone schooled in the old ways requires about as much material as is contained in two quarters of graduate school classes. As in the above example, the mathematics is simple, but there is a great deal of it, and few are patient enough to wade through it.

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